Imitation and Patent Licensing in a Cournot market

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October 2009

Abstract:

The present paper discusses and compares three different patent licensing regimes: a fixed fee, a per unit royalty and an auction in a homogeneous good Cournot market in which we find three competing firms where one of them is a patent holding firm. We suppose that innovative firm owns a process innovation reducing the marginal production cost by ε *. The key difference between the present model and models in the existing literature is that here we suppose the existence of an imitation risk of the patented innovation. In fact, we suppose that in situation of no licensing, a non innovative firm will imitate the new technology of the innovative firm with an imitation magnitude* γ *. We show that optimal licensing regime for an innovative firm depends on the magnitude of imitation compared with the magnitude of innovation. In fact, when innovation is not drastic compared with imitation, licensing by mean of royalties is better for the innovative firm while no licensing is the best strategy for an innovative firm when innovation is drastic compared with imitation. We also find that consumer surplus and total surplus are better with a licensing regime by mean of a fixed fee or an auction while a licensing regime by mean of a per unit royalty is the worst licensing strategy for consumers and total surplus.*

Keywords:
Technology transfer, process innovation, imitation

JEL classification:
C72 – D45 – L24 – O31 – O32 – O34

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1. Introduction

Firms apply for patents in the hopes of protecting their innovations against imitation by other competing firms. By patenting an innovation, an innovative firm benefits of a licensing revenue. The most known licensing regimes are a fixed fee which do not depend on the quantity produced, a per unit royalty depending on the quantity and an auction in which the patent holding firm sells a number of licenses to the highest price.

When innovative firm is active on the market, it can use its own innovation to produce at a lowest marginal cost. In fact, a process innovation allows to reduce the marginal cost of production and so to increase competition between firms. A patent holding firm will decide to license its innovation or not depending on its total revenue without licensing and with licensing. If the total revenue of an innovative firm when licensing is higher than its profit without licensing, the best strategy for a patent holding firm will be licensing. When the total revenue when licensing is lower than its profit without licensing, the best strategy will be to become a monopoly and to not share the new technology.

Authors in the literature have studied many models discussing technology transfer and comparing different patent licensing regimes according to magnitude of innovation and the position of the innovative firm: inside or outside of the market, the kind of innovation: process innovation or product innovation and have found many different results. The key difference between the present model and models in the existing literature is that here we suppose the existence of an imitation risk of the patented process innovation. We can find imitation in all industries. In fact, when there no licensing, non innovative firms will try to imitate the new technology with some magnitude which can depend on the power of imitation of a non innovative firm or the weakness or the strength of the innovation.

In this paper, we will study licensing strategies depending on the magnitude of imitation compared with the magnitude of innovation and the effects of each strategy on prices, consumer surplus and total surplus.

We show that when innovation is not drastic compared with imitation, royalties are better for an innovative firm. However, when innovation is drastic compared with imitation, the best strategy for an innovative firm is to become a monopoly and to not license its innovation.

We also show that consumer surplus and total surplus are better with a licensing by a mean of fixed fee or an auction and that royalties are the worst strategy for them.

2. Revue of literature

Many authors were interested by the intellectual property and patent licensing regimes. We can find different papers with different models and different results depending on many parameters like the position of the innovative firm: inside or outside of the market, the kind of competition: Cournot or Bertrand, kind of the produced goods: homogeneous or differentiated, the structure of the licensing contract: simple or combined, kind of innovation: process innovation or product innovation, kind of study: theoretical or empirical

Kamien and Tauman (1984, 1986), Katz and Shapiro (1986), Kamien and al (1992) focused on the position of the innovative firm. They find that when innovation is not drastic, royalties licensing is not better than licensing by a mean of fixed fee or an auction.

Wang (1998) studied a Cournot model in which the innovative firm is inside of the market and compared royalties and fixed fee licensing. He find that when innovation is not drastic a royalties licensing can be better that a fixed fee licensing. In other extension, Wang (2002) compared the two licensing modalities in a differentiated duopoly and find that royalties licensing is better than a fixed fee licensing for an innovative firm. He also finds that consumer surplus is better with fixed fee than royalties.

Giebe and Wolfstetter (2006) studied a combination between auction and royalties and find that when innovative firm is outside of the market, this combination is better that auction, royalties, fixed fee and combination between royalties and fixed fee. Sen (2002) show that, in a Cournot oligopoly, an innovative firm can realize its monopoly profit with a royalties licensing contract. Sen (2005), in an extension, show that when the number of licenses can take only integer values, royalties licensing can be better than fixed fee or auction licensing.

Caballero, Moner and Sempere (2002), show that consumer surplus is better with a fixed fee licensing. Sandonis and Fauli (2002) show that total surplus is lower with a Bertrand competition when innovation is enough drastic and for homogeneous goods.

In other papers, some authors compared the strategies of patenting an innovation and keeping the secret. Encaoua and Lefouili (2006) show that large innovations are likely to be kept secret whereas small innovations are always patented. They also show that, medium innovations are patented only when patent strength is sufficiently high.

Tauman and Aoki (2001) studied the licensing by mean of an auction in an oligopoly in presence of imitation and find that more the innovative firm sells licenses more non licensing firms become efficient.

Sohn (2008) compared strategies of imitation and innovation. He finds that imitation strategy is better when it allows a higher reduction of marginal cost. He also finds that imitation brake innovation but increase total surplus.

3. Games stages

Game consists in three stages. In the royalties and fixed fee licensing regimes, in the first stage, the patent holding firm choose a fixed fee or a per unit royalty maximizing its total revenue equal to the sum of its production profit and its licensing revenue. In the second stage, firms 2 and 3 decide to accept or not the licence offer of the patent holding firm. In the third and last stage, firms 1, 2 and 3 determine their equilibrium quantities in a Cournot competition. We suppose that non licensed firms will imitate the new patented technology and will benefit of an imitation reducing cost of γ while licensed firms benefit of a reducing cost of ε ($\gamma < \varepsilon$)

In the auction licensing, in the first stage, the patent holding firm will decide of the number of licenses. In the second stage, firms 2 and 3 decide simultaneously and independently of the auction amount. In the third and last stage, firms 1, 2 and 3 determine their equilibrium quantities in a Cournot competition.

4. Model

We consider 3 firms in a Cournot market producing a homogeneous good

- We find three firms on the market: the patent holding firm and two non innovative firms.
- Inverse demand function is:

  $p = a - bQ$, where p denotes the price and Q total output of the market
- With the old technology, the three firms produce at a constant unit cost c, $(0 < c < a)$
- The new technology of the patent holding firm allows to reduce the unit production cost of ε
- When they are not licensed, non innovative firms imitate the new technology and reduce their unit cost by γ $(0 < \gamma < \varepsilon)$. γ denotes the industrial spy parameter.
- We suppose that $\gamma \neq 0$ even when the patent holding firm do not license. We suppose that imitation risk is always present.

- Entry of new firms is not profitable

Cournot equilibrium

We consider Cournot equilibrium in a three firms market with production unit costs c_1, c_2 and c_3 respectively of firm 1, firm 2 and firm 3.

Equilibrium quantities are:

$$q_1* = \frac{a - 3c_1 + c_2 + c_3}{4b},\ q_2* = \frac{a - 3c_2 + c_1 + c_3}{4b},\ q_3* = \frac{a - 3c_3 + c_1 + c_2}{4b} \tag{1}$$

Equilibrium profits are

$$\pi_1* = \frac{(a - 3c_1 + c_2 + c_3)^2}{16b},\ \pi_2* = \frac{(a - 3c_2 + c_1 + c_3)^2}{16b},\ \pi_3* = \frac{(a - 3c_3 + c_1 + c_2)^2}{16b} \tag{2}$$

5. No licensing

We consider in this section that patent holding firm benefit alone of the new technology and do not license the process innovation to firms 2 and 3. Unit production cost of innovative firm will be $c_1 = c - \varepsilon$ while unit production costs of non innovative firms will be $c_2 = c_3 = c - \gamma$ since they imitate the patented new technology with a magnitude γ.

Using equations (1) and (2), we distinguish between two cases: drastic and non drastic innovation compared with imitation. A drastic innovation compared with imitation yield to a monopoly on the market. In fact, in this case, patent holding firm can sell its product at a price equal or lower than production unit costs of non licensed firms. Innovation is drastic compared with imitation when $\varepsilon \geq 2\gamma + a - c$ and non drastic when $\varepsilon < 2\gamma + a - c$.

Since $\gamma < \varepsilon < c$ we can rewrite drastic conditions as follows:

Innovation is drastic when $2\gamma + a - c \leq \varepsilon < c$ and it is non drastic when $\gamma < \varepsilon < 2\gamma + a - c$.

1.1. <u>Non drastic innovation compared with imitation</u>($\gamma < \varepsilon < 2\gamma + a - c$)

Cournot equilibrium outputs are:

$$q_1^{NL} = \frac{a-c+3\varepsilon-2\gamma}{4b},\ q^{NL} = q_2^{NL} = q_3^{NL} = \frac{a-c+2\gamma-\varepsilon}{4b} \tag{3}$$

Equilibrium profits are:

$$\pi_1^{NL} = \frac{(a-c+3\varepsilon-2\gamma)^2}{16b},\ \pi^{NL} = \pi_2^{NL} = \pi_3^{NL} = \frac{(a-c+2\gamma-\varepsilon)^2}{16b} \tag{4}$$

1.2. **Drastic innovation compared with imitation** ($2\gamma + a - c \le \varepsilon < c$)

Cournot equilibrium outputs are:

$$q_1^{NL} = \frac{a-c+\varepsilon}{2b},\ q^{NL} = q_2^{NL} = q_3^{NL} = 0 \tag{5}$$

Equilibrium profits are:

$$\pi_1^{NL} = \frac{(a-c+\varepsilon)^2}{4b},\ \pi^{NL} = \pi_2^{NL} = \pi_3^{NL} = 0 \tag{6}$$

6. Fixed fee licensing

Let's consider in this section that the licensing regime is a fixed fee. We assume that patent holding firm charges a fixed fee equal to F to firms 2 and 3 in exchange of a license.

Fixed fee F is not depending on the output produced by the firm buying the license. Since our model supposes the existence of two non innovative firms, we have to consider two cases:

- Only one license is sold and we denote by $\overline{\pi}, \overline{c}$ and $\overline{q}$ respectively profit, unit cost and the output of the licensed firm. We denote by $\underline{\pi}, \underline{c}$ and $\underline{q}$ respectively profit, unit cost and the output of the on licensed firm.
- Two licenses are sold and we denote by $\overline{\overline{\pi}}, \overline{\overline{c}}$ and $\overline{\overline{q}}$ respectively profit, unit cost and the output of the firms using the new technology.

6.1. Only one license is sold

Production unit costs are: $c_1 = \bar{c} = c - \varepsilon$, and $\underline{c} = c - \gamma$

We obtain:

$$q_1 = \bar{q} = \frac{a - c + 2\varepsilon - \gamma}{4b},\ \underline{q} = \frac{a - c + 3\gamma - 2\varepsilon}{4b} \tag{13}$$

$$\pi_1 = \bar{\pi} = \frac{(a - c + 2\varepsilon - \gamma)^2}{16b},\ \underline{\pi} = \frac{(a - c + 3\gamma - 2\varepsilon)^2}{16b} \tag{14}$$

6.1.1. If $\gamma < \varepsilon < \frac{3\gamma + a - c}{2}$

Cournot equilibrium Outputs are:

$$q_1 = \bar{q} = \frac{a - c + 2\varepsilon - \gamma}{4b},\ \underline{q} = \frac{a - c + 3\gamma - 2\varepsilon}{4b} \tag{13}$$

Cournot equilibrium profits are:

$$\pi_1 = \bar{\pi} = \frac{(a - c + 2\varepsilon - \gamma)^2}{16b},\ \underline{\pi} = \frac{(a - c + 3\gamma - 2\varepsilon)^2}{16b} \tag{14}$$

6.1.2. If $\frac{3\gamma + a - c}{2} \leq \varepsilon < c$

Cournot equilibrium Outputs are:

$$q_1 = \bar{q} = \frac{a - c + \varepsilon}{3b},\ \underline{q} = 0 \tag{15}$$

Cournot equilibrium profits are:

$$\pi_1 = \bar{\pi} = \frac{(a - c + \varepsilon)^2}{9b},\ \underline{\pi} = 0 \tag{16}$$

If $\gamma < \varepsilon < \dfrac{3\gamma + a - c}{2}$ **(non drastic innovation)**

Using equations (14) and (4):

(17)

$$F = \bar{\pi} - \pi^{NL} = \frac{3(2a\varepsilon - 2a\gamma - 2c\varepsilon + 2c\gamma + \varepsilon^2 - \gamma^2)}{16b}$$

Total revenue of the patent holding firm is (using (17) and (14))

(18)

$$\pi_1 + F = \left[\frac{(a - c + 2\varepsilon - \gamma)^2}{16b} + \frac{3(2a\varepsilon - 2a\gamma - 2c\varepsilon + 2c\gamma + \varepsilon^2 - \gamma^2)}{16b}\right]$$

$$RT_1 = \pi_1 + F = \frac{a^2 - 2ac + 10a\varepsilon - 8a\gamma + c^2 - 10c\varepsilon + 8c\gamma + 7\varepsilon^2 - 4\varepsilon\gamma - 2\gamma^2}{16b}$$

Comparing equations (18) and (4) we find

$\pi_1 + F > \pi_1^{NL}$ (Appendix 1)

When $\gamma < \varepsilon < \dfrac{3\gamma + a - c}{2}$**,** the patent holding firm has interest to license only one firm because its profit is higher than the non licensing regime.

If $\dfrac{3\gamma + a - c}{2} < \varepsilon < c$

We distinguish between two cases: $\dfrac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ (non drastic innovation) and $2\gamma + a - c < \varepsilon < c$ (drastic innovation)

6.1.2.1. $\dfrac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ **(non drastic innovation)**

Using equations (16) and (4) :

(19)

$$F = \bar{\pi} - \pi^{NL} = \frac{7a^2 - 14ac + 50a\varepsilon + 7c^2 - 50c\varepsilon + 7\varepsilon^2 - 36a\gamma + 36c\gamma - 36\gamma^2 + 36\gamma\varepsilon}{144b}$$

Total revenue of the patent holding firm using equations (19) and (16) is :

(20)

$$\pi_1 + F = \frac{23a^2 - 46ac + 82a\varepsilon + 23c^2 - 82c\varepsilon + 23\varepsilon^2 - 36a\gamma + 36c\gamma - 36\gamma^2 + 36\varepsilon\gamma}{144b}$$

Comparing equations (20) and (4), we find:

$\pi_1 + F > \pi_1^{PL}$ if $\frac{3\gamma + a - c}{2} < \varepsilon < 1{,}788\gamma + 0{,}788(a-c)$ (Appendix 2)

$\pi_1 + F < \pi_1^{PL}$ if $1{,}788\gamma + 0{,}788(a-c) < \varepsilon < 2\gamma + a - c$ (Appendix 2)

When $\frac{3\gamma + a - c}{2} < \varepsilon < 1{,}788\gamma + 0{,}788(a-c)$, patent holding firm has interest to license only one firm because its profit is higher than the non licensing regime.

When $1{,}788\gamma + 0{,}788(a-c) < \varepsilon < 2\gamma + a - c$ on licensing is better for patent holding firm than licensing only one firm.

6.1.2.2. $2\gamma + a - c < \varepsilon < c$ (drastic innovation)

Using equations (16) and (6):

(21)

$$F = \bar{\pi} - \pi^{NL} = \frac{(a - c + \varepsilon)^2}{9b}$$

Total revenue of the patent holding firm, using equations (21) and (16), is:

(22)

$$\pi_1 + F = \frac{2(a - c + \varepsilon)^2}{9b}$$

Comparing equations (22) and (6), we find:

$\pi_1 + F < \pi_1^{NL}$

Since $RT_1^{1F} - \pi_1^{NL} = \frac{-(a - c + \varepsilon)^2}{36b} < 0$

Patent holding firm has not interest in licensing since its profit is lower than the non licensing regime.

6.2. Two licenses are sold

Production unit costs are: $c_1 = \bar{\bar{c}} = c - \varepsilon$. Replacing in equations (1) and (2), we obtain:

(23)

$$q_1 = \overline{\overline{q}} = \frac{a-c+\varepsilon}{4b}$$

(24)

$$\pi_1 = \overline{\overline{\pi}} = \frac{(a-c+\varepsilon)^2}{16b}$$

$$F = \overline{\overline{\pi}} - \pi_2^{NL}$$

We have to consider two cases since π_2^{NL} depends on the magnitude of innovation ε compared with imitation γ.

In fact, for $\gamma < \varepsilon < 2\gamma + a - c$, we have $\pi^{NL} = \pi_2^{NL} = \pi_3^{NL} = \dfrac{(a-c+2\gamma-\varepsilon)^2}{16b}$ and for $2\gamma + a - c \le \varepsilon < c$ we have $\pi^{NL} = \pi_2^{NL} = \pi_3^{NL} = 0$.

6.2.1. $\gamma < \varepsilon < 2\gamma + a - c$ (non drastic innovation)

Using equations (24) and (14) we find:

(25)

$$F = \overline{\overline{\pi}} - \pi_2^{NL} = \frac{(\varepsilon-\gamma)(a-c+\gamma)}{4b}$$

Patent holding firm total revenue, using equations (25) and (24) is:

(26)

$$\pi_1 + 2F = \frac{a^2 - 2ac + 10a\varepsilon + c^2 - 10c\varepsilon + \varepsilon^2 - 8a\gamma + 8c\gamma - 8\gamma^2 + 8\gamma\varepsilon}{16b}$$

Comparing equations (26) and (4) we find (Appendix 3):

$$\pi_1 + 2F > \pi_1^{NL} \text{ if } \gamma < \varepsilon < \frac{3\gamma + a - c}{2}$$

$$\pi_1 + 2F < \pi_1^{NL} \text{ if } \frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$$

When $\gamma < \varepsilon < \frac{3\gamma + a - c}{2}$, patent holding firm has interest to license firms 2 and 3 since its profit is higher than the non licensing regime. When $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$, the non licensing strategy is better than the two licenses regime for the patent holding firm.

6.2.2. $2\gamma + a - c \le \varepsilon < c$

Using equations (24) and (16), we find:

$$F = \overline{\overline{\pi}} - \pi_2^{NL} = \frac{(a - c + \varepsilon)^2}{16b} \tag{27}$$

Using equations (27) and (24), we compute total revenue of the patent holding firm:

$$\pi_1 + 2F = \frac{3(a - c + \varepsilon)^2}{16b} \tag{28}$$

Comparing equations (28) and (4) we find:

$\pi_1 + 2F < \pi_1^{NL}$

In fact, $RT_1 - \pi_1^{NL} = -\frac{(a - c + \varepsilon)^2}{16b} < 0$

The patent holding firm has not interest to license the two firms and would better to sell no license when $2\gamma + a - c \le \varepsilon < c$ since no licensing regime profit is higher than its total revenue in the two license regime.

6.3. Comparison between fixed fee regimes of one license and two licenses

We distinguish between three cases: $\gamma < \varepsilon < \frac{3\gamma + a - c}{2}$ (non drastic innovation), $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ (non drastic innovation) and $2\gamma + a - c < \varepsilon < c$ (drastic innovation)

- **If** $\gamma < \varepsilon < \frac{3\gamma + a - c}{2}$

Comparing total revenues of the patent holding firm in regimes of one license and two licenses (using equations (26) and (18)), we find:

$RT_1^{2F} < RT_1^{1F}$ (Appendix 4)

- **If** $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$

Comparing total revenues of the patent holding firm in regimes of one license and two licenses (using equations (26) and (20)), we find:

$RT_1^{2F} < RT_1^{1F}$ (Appendix 5)

- **If** $2\gamma + a - c < \varepsilon < c$

Comparing total revenues of the patent holding firm in regimes of one license and two licenses (using equations (28) and (22)), we find:

$RT_1^{2F} < RT_1^{1F}$ (Appendix 6)

6.3.1. Comparison of the patent holding firm profits

a) $\gamma < \varepsilon < \frac{3\gamma + a - c}{2}$ **(non drastic innovation)**

Comparing patent holding firm total revenue for the three fixed fee licensing strategies (one license, two licenses and no license) we find that $RT_1^{1F} > RT_1^{2F} > \pi_1^{PL}$.

Patent holding firm has interest to sell only one license.

b) $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ **(non drastic innovation)**

Comparing patent holding firm total revenue, we find

- $RT_1^{1F} > \pi_1^{PL} > RT_1^{2F}$ when $\frac{3\gamma + a - c}{2} < \varepsilon < 1{,}788\gamma + 0{,}788(a - c)$**:** Patent holding firm has interest to sell only one license.
- $\pi_1^{PL} > RT_1^{1F} > RT_1^{2F}$ when $1{,}788\gamma + 0{,}788(a - c) < \varepsilon < 2\gamma + a - c$**:** Patent holding firm has interest to sell no license.

c) $2\gamma + a - c < \varepsilon < c$ **(drastic innovation)**

Patent holding firm do not license its innovation since its profit is higher than the one license and the two licenses fixed fee regimes.

$\pi_1^{PL} > RT_1^{1F} > RT_1^{2F}$

Result 1

For a fixed fee regime,

- $\gamma < \varepsilon < \dfrac{3\gamma + a - c}{2}$ **(non drastic innovation)**

(17)

Fixed fee is $F = \dfrac{3(2(\varepsilon - \gamma)(a-c) + \varepsilon^2 - \gamma^2)}{16b}$, only one license is sold

(18)

Total revenue of patent holding firm is

$$\frac{a^2 - 2ac + 10a\varepsilon - 8a\gamma + c^2 - 10c\varepsilon + 8c\gamma + 7\varepsilon^2 - 4\varepsilon\gamma - 2\gamma^2}{16b}$$

- $\dfrac{3\gamma + a - c}{2} < \varepsilon < 1{,}788\gamma + 0{,}788(a-c)$ **(non drastic innovation)**

(19)

Fixed fee is $F = \dfrac{7a^2 - 14ac + 50a\varepsilon + 7c^2 - 50c\varepsilon + 7\varepsilon^2 - 36a\gamma + 36c\gamma - 36\gamma^2 + 36\gamma\varepsilon}{144b}$, only one license is sold.

Total revenue of patent holding firm is

(20)

$$\frac{23a^2 - 46ac + 82a\varepsilon + 23c^2 - 82c\varepsilon + 23\varepsilon^2 - 36a\gamma + 36c\gamma - 36\gamma^2 + 36\varepsilon\gamma}{144b}$$

- $1{,}788\gamma + 0{,}788(a-c) < \varepsilon < 2\gamma + a - c$ **(non drastic innovation)**

Patent holding firm do not license its innovation.

(4)

Patent holding firm profit is $\pi_1^{PL} = \dfrac{(a - c + 3\varepsilon - 2\gamma)^2}{16b}$

- $2\gamma + a - c < \varepsilon < c$ **(drastic innovation)**

(6)

Patent holding firm do not license its innovation and its monopoly profit is $\pi_1 = \frac{(a-c+\varepsilon)^2}{4b}$

6.3.2. Surplus comparison :

Comparing consumer surplus and total surplus we find that they are higher when patent holding firm sell two licenses (Appendix 7, 8, 9)

7. Royalties licensing regime

We consider here that the licensing regime is a per unit royalty. Let's denote by r the per unit royalty charged by the patent holding firm to the licensed firms. We suppose that the per unit royalty $r \in [0, \varepsilon]$ unless non innovative firms will not buy licenses. Let's denote by $\overset{=}{\pi}$, $\overset{=}{c}$ and $\overset{=}{q}$ respectively the profit, unit cost and output of firms using the new technology. Unit costs are: $c_1 = c - \varepsilon$, and $\overset{=}{c} = c - \varepsilon + r$.

From equations (1) and (2), Cournot equilibrium outputs are:

$$q_1 = \frac{a-c+\varepsilon+2r}{4b}, \ \overset{=}{q} = \frac{a-c+\varepsilon-2r}{4b} \tag{29}$$

Cournot equilibrium profits are:

$$\pi_1 = \frac{(a-c+\varepsilon+2r)^2}{16b}, \ \overset{=}{\pi} = \frac{(a-c+\varepsilon-2r)^2}{16b} \tag{30}$$

Patent holding firm total revenue:

$$\pi_1 + 2r\overset{=}{q} = \left[\frac{(a-c+\varepsilon+2r)^2}{16b} + 2r \frac{(a-c+\varepsilon-2r)}{4b} \right] \tag{31}$$

(Appendix 10)

7.1. If $\gamma < \varepsilon < a - c$

Computing the optimal value of the per unit royalty $r = \varepsilon$ in equations (29) and (30), we find the following equilibrium outputs and profits of the three competing firms:

(32)

$$q_1 = \frac{a-c+3\varepsilon}{4b}, \ \overset{=}{q} = \frac{a-c-\varepsilon}{4b}$$

(33)

$$\pi_1 = \frac{(a-c+3\varepsilon)^2}{16b}, \ \overset{=}{\pi} = \frac{(a-c-\varepsilon)^2}{16b}$$

Total revenue of patent holding firm, using equation (31):

(34)

$$\pi_1 + 2r\overset{=}{q} = \left[\frac{(a-c+3\varepsilon)^2}{16b} + \frac{\varepsilon(a-c-\varepsilon)}{2b}\right] = \frac{(a-c)^2 + 14\varepsilon(a-c) + \varepsilon^2}{16b}$$

Consumer surplus is $SC = \frac{(3(a-c)+\varepsilon)^2}{32b}$

Total surplus is $ST = \frac{15(a-c)^2 + 10\varepsilon(a-c) + 23\varepsilon^2}{32b}$

7.2. If $\varepsilon \geq a-c$

Computing the optimal value of the per unit royalty $r = \frac{a-c+\varepsilon}{2}$ in equations (29) and (30) we find:

(35)

$$q_1 = \frac{a-c+\varepsilon}{2b}, \ \overset{=}{q} = 0$$

(36)

$$\pi_1 = \frac{(a-c+\varepsilon)^2}{4b}, \ \overset{=}{\pi} = 0$$

Using equation (31), total revenue of patent holding firm equals:

(37)

$$\pi_1 + 2r\overset{=}{q} = \left[\frac{(a-c+\varepsilon)^2}{4b} + 0\right] = \frac{(a-c+\varepsilon)^2}{4b} = \pi_1^{NL}$$

Consumer surplus is $SC = \frac{(a-c+\varepsilon)^2}{8b}$

Total surplus is $ST = \frac{3(a-c+\varepsilon)^2}{8b}$

7.3. Comparison

- **If** $\gamma < \varepsilon < a-c$

  - $RT_{2\,firms}^{royalties} > RT^{NL}$

According to equations (34) and (4), licensing the two firms in the royalty regime is better than no licensing for the patent holding firm (Appendix 11).

  - $SC_{2\,firms}^{royalties} < SC^{NL}$

No licensing is better for consumers than royalties licensing since $SC_{2\,firms}^{royalties} - SC^{NL} = \frac{-\gamma(3(a-c)+\varepsilon+\gamma)}{8b} < 0$

  - $ST_{2\,firms}^{royalties} < ST^{NL}$ if $\gamma < \varepsilon < \frac{5(a-c)+7\gamma}{9}$ and $ST_{2\,firms}^{royalties} > ST^{NL}$ if $\frac{5(a-c)+7\gamma}{9} < \varepsilon < a-c$

When innovation is not drastic compared with imitation, total surplus is higher under no licensing regime. $ST_{2\,firms}^{royalties} - ST^{NL} = \frac{-\gamma(5(a-c)-9\varepsilon+7\gamma)}{8b}$

- **If** $a-c < \varepsilon < 2\gamma + a - c$

  - $RT_{2\,firms}^{royalties} > RT^{NL}$

Using equations (37) and (4), we find that licensing by royalties is better than no licensing for the patent holding firm since its profit is higher with royalties licensing regime.

In fact, $RT_{2\,firms}^{royalties} - RT^{NL} = \frac{(3(a-c)+5\varepsilon-2\gamma)(a-c-\varepsilon+2\gamma)}{16b} > 0$

  - $SC_{2\,firms}^{royalties} < SC^{NL}$

Consumer surplus is better under no licensing regime.

$$SC_{2\,firms}^{royalties} - SC^{NL} = -\frac{(a-c-\varepsilon+2\gamma)(5(a-c)+3\varepsilon+2\gamma)}{32b} < 0$$

- $ST_{2\,firms}^{royalties} > ST^{NL}$ if $a-c<\varepsilon<2\gamma+a-c$ and if $\varepsilon > \dfrac{3(a-c)+14\gamma}{11}$

  $ST_{2\,firms}^{royalties} < ST^{NL}$ if $a-c<\varepsilon<2\gamma+a-c$ and if $\varepsilon < \dfrac{3(a-c)+14\gamma}{11}$

Total surplus is better under no licensing regime when innovation is not drastic compared with imitation since $ST_{2\,firms}^{royalties} - ST^{NL} = -\dfrac{(a-c-\varepsilon+2\gamma)(3(a-c-11\varepsilon+14\gamma)}{32b}$

- **If** $\varepsilon > 2\gamma + a - c$

When innovation is too drastic compared with imitation, royalties and no licensing regimes are equivalent for the patent holding firm, consumers and total surplus (Appendix 12).

Proposition 1

When $\varepsilon < a-c$ (non drastic innovation), optima royalty rate is $r^* = \varepsilon$ and the two firms are licensed. When $a-c<\varepsilon<2\gamma+a-c$ (drastic innovation), optimal royalty rate is $r^* = \dfrac{a-c+\varepsilon}{2}$ and the two firms realize profits equal to zero. When $\varepsilon \geq 2\gamma+a-c$ (drastic innovation), patent holding firm do not license its innovation and become a monopoly.
Comparing previous results, we can say that the patent holding firm has interest to license its innovation under royalties regime when innovation is not drastic compared with imitation. Concerning total surplus, we find that when innovation is not licensed and when it is not drastic compared with imitation or too drastic, total surplus is higher. When innovation is intermediate compared with imitation, total surplus is better when innovation is licensed.

8. Auction licensing

We consider in this section that innovation is sold in exchange of an auction. We denote by E the maximum amount that firms 2 and 3 can pay to buy a license.

Two cases follow:

- The license is sold to only one firm. We denote by $\bar{\pi}, \bar{c}$ and $\bar{q}$ respectively the profit, unit cost and output of the licensed firm and by $\underline{\pi}, \underline{c}$ and $\underline{q}$ respectively the profit, unit cost and output of the non licensed firms.
- The license is sold to two firms. We denote by $\bar{\bar{\pi}}, \bar{\bar{c}}$ and $\bar{\bar{q}}$ the profit, unit cost and output of the firms using the new technology.

8.1. only one license is sold

Production unit costs are $c_1 = \bar{c} = c - \varepsilon$ and $\underline{c} = c - \gamma$. Using equations (1) and (2) we obtain:

$$q_1 = \bar{q} = \frac{a - c + 2\varepsilon - \gamma}{4b}, \; \underline{q} = \frac{a - c - 2\varepsilon + 3\gamma}{4b} \tag{38}$$

$$\pi_1 = \bar{\pi} = \frac{(a - c + 2\varepsilon - \gamma)^2}{16b}, \; \underline{\pi} = \frac{(a - c - 2\varepsilon + 3\gamma)^2}{16b} \tag{39}$$

- **If** $\varepsilon < \frac{3\gamma + a - c}{2}$

Cournot equilibrium outputs are:

$$q_1 = \bar{q} = \frac{a - c + 2\varepsilon - \gamma}{4b}, \; \underline{q} = \frac{a - c - 2\varepsilon + 3\gamma}{4b} \tag{38}$$

Cournot equilibrium profits are:

$$\pi_1 = \bar{\pi} = \frac{(a - c + 2\varepsilon - \gamma)^2}{16b}, \; \underline{\pi} = \frac{(a - c - 2\varepsilon + 3\gamma)^2}{16b} \tag{39}$$

- **If** $\varepsilon \geq \frac{3\gamma + a - c}{2}$

Cournot equilibrium outputs are:

(40)

$$q_1 = \bar{q} = \frac{a-c+\varepsilon}{3b},\ \underline{q} = 0$$

Cournot equilibrium profits are:

(41)

$$\pi_1 = \bar{\pi} = \frac{(a-c+\varepsilon)^2}{9b},\ \underline{\pi} = 0$$

Since innovation is drastic if $\varepsilon \geq 2\gamma + a - c$ and non drastic if $\varepsilon < 2\gamma + a - c$, we will distinguish between two cases:

8.1.1. $\varepsilon < \frac{3\gamma + a - c}{2}$ (Non drastic innovation)

Using equations (39) we compute the amount of the auction:

(42)

$$E = \bar{\pi} - \underline{\pi} = \frac{(a-c+\gamma)(\varepsilon-\gamma)}{2b}$$

The patent holding firm total revenue, using equations (39) and (42), is:

(43)

$$\pi_1 + E = \left[\frac{(a-c+2\varepsilon-\gamma)^2}{16b} + \frac{(a-c+\gamma)(\varepsilon-\gamma)}{2b}\right]$$

$$= \frac{(a-c)^2 + 4\varepsilon^2 + 12\varepsilon(a-c) - 10\gamma(a-c) + 4\varepsilon\gamma - 7\gamma^2}{16b}$$

Comparing equations (43) and (4) we find $\pi_1 + E > \pi_1^{NL}$ (Appendix 13)

When $\varepsilon < \frac{3\gamma + a - c}{2}$**,** licensing one firm is better for the patent holding firm than no licensing since its total revenue is higher.

8.1.2. $\varepsilon > \frac{3\gamma + a - c}{2}$

We distinguish here between two cases : $\frac{3\gamma + a - c}{2} \leq \varepsilon < 3\gamma + a - c$ (non drastic innovation) and $\varepsilon \geq 3\gamma + a - c$ (drastic innovation)

8.1.2.1. $\frac{3\gamma + a - c}{2} \leq \varepsilon < 2\gamma + a - c$ **(non drastic innovation)**

Using equations (41) we find:

(44)

$$E = \overline{\pi} - \underline{\pi} = \frac{(a - c + \varepsilon)^2}{9b}$$

Total revenue of the patent holding firm is:

(45)

$$\pi_1 + E = \frac{2(a - c + \varepsilon)^2}{9b}$$

Using equations (45) and (4) we find:

If $\frac{3\gamma + a - c}{2} \leq \varepsilon < 1{,}794\gamma + 0{,}7947(a - c)$ then $\pi_1 + E - \pi_1^{NL} > 0 \Leftrightarrow RT_1^{1E} > \pi_1^{NL}$

If $1{,}794\gamma + 0{,}7947(a - c) \leq \varepsilon < 2\gamma + a - c$ then $\pi_1 + E - \pi_1^{NL} < 0 \Leftrightarrow RT_1^{1E} < \pi_1^{NL}$

(Appendix 14)

The patent holding firm has interest to license one firm, when $\frac{3\gamma + a - c}{2} \leq \varepsilon < 1{,}794\gamma + 0{,}7947(a - c)$, since its profit is higher then non licensing. However, when $1{,}794\gamma + 0{,}7947(a - c) \leq \varepsilon < 2\gamma + a - c$, its non licensing profit is better then total revenue of one firm licensing.

8.1.2.2. $\varepsilon \geq 2\gamma + a - c$ **(drastic innovation)**

The amount of auction, using equations (41) is:

(44)

$$E = \overline{\pi} - \underline{\pi} = \frac{(a - c + \varepsilon)^2}{9b}$$

Using equations (41) and (44), total profit of the patent holding firm is:

(45)

$$\pi_1 + E = \frac{2(a - c + \varepsilon)^2}{9b}$$

Comparing equations (45) and (6), we find: $\pi_1 + E < \pi_1^{NL}$

It appears that the patent holding firm has not interest to license one firm since no licensing is better than licensing.

In fact, $\pi_1 + E - \pi_1{}^{NL} = -\frac{(a-c+\varepsilon)^2}{36b} < 0$

8.2. license sold to the two firms

Production unit costs are $c_1 = \overline{\overline{c}} = c - \varepsilon$. Replacing in equations (1) and (2), we obtain:

(46)

$$q_1 = \overline{\overline{q}} = \frac{a-c+\varepsilon}{4b}$$

(47)

$$\pi_1 = \overline{\overline{\pi}} = \frac{(a-c+\varepsilon)^2}{16b}$$

$$E = \overline{\overline{\pi}} - \underline{\pi}$$

We have to consider two cases since $\underline{\pi}$ is depending on the magnitude of the innovation ε compared with imitation γ.

If $\varepsilon < \frac{3\gamma + a - c}{2}$ we have $\underline{\pi} = \frac{(a-c-2\varepsilon+3\gamma)^2}{16b}$, and when $\varepsilon > \frac{3\gamma + a - c}{2}$ we have $\underline{\pi} = 0$.

8.2.1. $\varepsilon < \frac{3\gamma + a - c}{2}$ (non drastic innovation)

Using equations (47) and (39) the amount of auction is:

(48)

$$E = \overline{\overline{\pi}} - \underline{\pi} = \frac{3(\varepsilon-\gamma)(-\varepsilon-2c+2a+3\gamma)}{16b}$$

Using equations (48) and (47), total revenue of the patent holding firm is:

(49)

$$\pi_1 + 2E = \left[\frac{(a-c+\varepsilon)^2}{16b} + 2\frac{3(\varepsilon-\gamma)(-\varepsilon-2c+2a+3\gamma)}{16b} \right]$$

$$= \frac{(a-c)^2 - 5\varepsilon^2 + 14\varepsilon(a-c) - 12\gamma(a-c) + 24\varepsilon\gamma - 18\gamma^2}{16b}$$

Comparing equations (49) and (4) we find: $\pi_1 + 2E > \pi_1^{NL}$ (Appendix 15)

Licensing two firms is better for patent holding firm than no licensing when $\varepsilon < \frac{3\gamma + a - c}{2}$ since its profit is higher

8.2.2. $\varepsilon > \frac{3\gamma + a - c}{2}$

We distinguish here between two cases: $\frac{3\gamma + a - c}{2} \le \varepsilon < 3\gamma + a - c$ (non drastic innovation) and $\varepsilon \ge 3\gamma + a - c$ (drastic innovation)

8.2.2.1. $\frac{3\gamma + a - c}{2} \le \varepsilon < 2\gamma + a - c$ **(non drastic innovation)**

Using equations (47) and (41), we find:

(50)

$$E = \overline{\overline{\pi}} - \underline{\pi} = \frac{(a-c+\varepsilon)^2}{16b}$$

Using equations (50) and (47), patent holding firm total revenue is:

(51)

$$\pi_1 + 2E = \frac{3(a-c+\varepsilon)^2}{16b}$$

Comparing equations (51) and (4), we find:

If $\frac{3\gamma + a - c}{2} \le \varepsilon < 1{,}5773\gamma + 0{,}5773(a-c)$ then $\pi_1 + 2E - \pi_1^{NL} > 0 \Leftrightarrow RT_1^{1E} > \pi_1^{NL}$

If $1{,}5773\gamma + 0{,}5773(a-c) \le \varepsilon < 2\gamma + a - c$ then $\pi_1 + 2E - \pi_1^{NL} < 0 \Leftrightarrow RT_1^{1E} < \pi_1^{NL}$

(Appendix 16)

When $\frac{3\gamma + a - c}{2} \leq \varepsilon < 1{,}5773\gamma + 0{,}5773(a-c)$, licensing one firm is better than no licensing. However, when $1{,}5773\gamma + 0{,}5773(a-c) \leq \varepsilon < 2\gamma + a - c$ no licensing is better than licensing one firm.

8.2.2.2. $\varepsilon \geq 2\gamma + a - c$ (drastic innovation)

Using equations (47) and (41) we find:

$$E = \overline{\overline{\pi}} - \underline{\pi} = \frac{(a-c+\varepsilon)^2}{16b} \tag{50}$$

Total revenue of patent holding firm, using equations (50) and (47), is:

$$\pi_1 + 2E = \frac{3(a-c+\varepsilon)^2}{16b} \tag{51}$$

Comparing equations (51) and (6), we find: $\pi_1 + 2E < \pi_1^{NL}$

No licensing is better than licensing the innovation to two firms since the no licensing profit of the patent holding firm is higher. In fact $\pi_1 + 2E - \pi_1^{NL} = -\frac{(a-c+\varepsilon)^2}{16b} < 0$

Proposition 2:
Selling two licenses is better for the patent holding firm than selling only one license when innovation is not drastic compared with imitation ($\gamma < \varepsilon < \frac{11\gamma + 2(a-c)}{9}$) since its profit is higher. When innovation is enough or too drastic compared with imitation ($\varepsilon > \frac{11\gamma + 2(a-c)}{9}$), selling only one license is better for the patent holding firm (Appendix 17).

Result 2

For an auction licensing,

- $\varepsilon < \dfrac{11\gamma + 2(a-c)}{9}$ **(non drastic innovation)**

(48)

The auction is $E = \dfrac{(\varepsilon - \gamma)(2(a-c) - \varepsilon + 3\gamma)}{16b}$, license is sold to two firms.

- $\dfrac{11\gamma + 2(a-c)}{9} < \varepsilon < \dfrac{3\gamma + a - c}{2}$ **(non drastic innovation)**

(42)

The auction is $E = \dfrac{(\varepsilon - \gamma)(a - c + \gamma)}{2b}$, license is sold to only one firm.

- $\dfrac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ **(innovation non intense)**

(44)

The auction is $E = \dfrac{(a - c + \varepsilon)^2}{9b}$, license is sold to only one firm.

(45)

Total revenue of patent holding firm is $TR_1 = \dfrac{2(a - c + \varepsilon)^2}{9b}$

- $\varepsilon \geq 2\gamma + a - c$ **(drastic innovation)**

Patent holding firm do not license and become a monopoly.

(6)

Patent holding firm profit is $\dfrac{(a - c + \varepsilon)^2}{4b}$

Proposition 3

Consumer surplus and total surplus are higher when the two firms benefit of a license. Licensing only one firm is better than no licensing for consumers and society (Appendix 18)

9. Optimal licensing regime for patent holding firm

9.1. If $\gamma < \varepsilon < \frac{11\gamma + 2(a-c)}{9}$

Royalties licensing is better for the patent holding firm. The worst licensing regime is no licensing. In fact, comparing equations (34), (37), (49), (43), (18), (26) and (4) we find:

$RT_1^{Royalties} > RT_1^{2E} > RT_1^{1E} > RT_1^{1F} > RT_1^{2F} > RT_1^{NL}$ (Appendix 19)

9.2. If $\frac{11\gamma + 2(a-c)}{9} < \varepsilon < \frac{3\gamma + a - c}{2}$

Royalties licensing is better than auction licensing (Appendix 20). Comparing total revenue of the patent holding firm for other regimes, we find:

If $\frac{11\gamma + 2(a-c)}{9} < \varepsilon < \frac{4\gamma + a - c}{3}$ than

$RT_1^{Royalties} > RT_1^{1E} > RT_1^{2E} > RT_1^{1F} > RT_1^{2F} > RT_1^{NL}$ (Appendix 21)

If $\frac{4\gamma + a - c}{3} < \varepsilon < \frac{3\gamma + a - c}{2}$ than

$RT_1^{Royalties} > RT_1^{1E} > RT_1^{1F} > RT_1^{2E} > RT_1^{2F} > RT_1^{NL}$ (Appendix 21)

9.3. If $\frac{3\gamma + a - c}{2} < \varepsilon < 1{,}788\gamma + 0{,}788(a-c)$

Royalties licensing is the best regime. While fixed fee licensing to two firms is the worst licensing regime.

$RT_1^{Royalties} > RT_1^{1E} > RT_1^{1F} > RT_1^{2E} > RT_1^{NL} > RT_1^{2F}$ (Appendix 22).

9.4. If $1{,}788\gamma + 0{,}788(a-c) < \varepsilon < 2\gamma + a - c$

Royalties licensing is the best regime. While no licensing is better than fixed fee licensing

$RT_1^{Royalties} > RT_1^{1E} > RT_1^{2E} > RT_1^{NL} > RT_1^{1F} > RT_1^{2F}$ (Appendix 23)

9.5. If $2\gamma + a - c < \varepsilon < c$

Comparing equations (37), (6), (45), (22), (51) and (28), we find

$\left\{RT_1^{Royalties} = RT_1^{NL}\right\} > \left\{RT_1^{1E} = RT_1^{1F}\right\} > \left\{RT_1^{2E} = RT_1^{2F}\right\}$

The patent holding firm has interest to not license its innovation (Appendix 24).

Proposition 4

Patent holding firm has interest to license by royalties regime when innovation is not drastic or intermediate compared with imitation ($\gamma < \varepsilon < 2\gamma + a - c$). When innovation is drastic ($2\gamma + a - c < \varepsilon < c$), the optimal regime for the patent holding firm is no licensing. For equal licenses, auction is better than fixed fee licensing.

When patent holding firm has to choose between fixed fee or auction licensing, the optimal strategy maximizing its total revenue is to license the two firms by an auction when innovation is not drastic ($\gamma < \varepsilon < \frac{11\gamma + 2(a-c)}{9}$) and to license only one firm by auction when innovation is intermediate compared with imitation ($\frac{11\gamma + 2(a-c)}{9} < \varepsilon < 2\gamma + a - c$). When imitation is drastic, the best strategy is no licensing and becoming a monopoly.

10. Optimal regime for consumers in terms of the price

- If $\gamma < \varepsilon < 2\gamma + a - c$ (non drastic innovation)

The higher price appears in the royalties licensing regime $p^{royalties} = \frac{a + 3c - \varepsilon}{4}$. We show that the output price of no licensing is lower than royalties licensing $p^{NL} = \frac{a + 3c - \varepsilon - 2\gamma}{4}$. We also find that the output price in fixed fee or auction licensing to only one firm is higher than the price of no licensing regime. $p^{1E} = p^{1F} = \frac{a + 3c - 2\varepsilon - \gamma}{4}$. The lowest output price is obtained in auction or fixed fee licensing for two firms $p^{2E} = p^{2F} = \frac{a + 3c - 3\varepsilon}{4}$.

- If $2\gamma + a - c < \varepsilon < c$

When innovation is drastic compared with imitation, the highest output price is obtained in no licensing or royalties licensing $p^{NL} = p^{royalties} = \frac{a + c - \varepsilon}{2}$. However, auction or fixed fee licensing to two firms are the best for consumers n the point of view of the price of the output equal to $p^{2E} = p^{2F} = \frac{a + 3c - 3\varepsilon}{4}$.

Proposition 5

Output price is lower when patent holding firm license its innovation to the two firms by a fixed fee or an auction and it is equal to $p^{2E} = p^{2F} = \frac{a+3c-3\varepsilon}{4}$. The price is higher when the patent holding firm become a monopoly and for a drastic innovation compared with imitation and equals to $p^{NL} = \frac{a+c-\varepsilon}{2}$. Price is higher when licensing is by mean of a royalty when innovation is non drastic ($\gamma < \varepsilon < a-c$) with a royalty output price equal to $p^{royalties} = \frac{a+3c-\varepsilon}{4}$ or when innovation is too drastic ($a-c < \varepsilon < 2\gamma + a - c$) with an input price equal to $p^{royalties} = \frac{a+c-\varepsilon}{2}$.

11. Optimal licensing regime for consumers in terms of surplus

We find that consumer surplus for non drastic and drastic innovations compared with imitation ($\gamma < \varepsilon < 2\gamma + a - c$ or $2\gamma + a - c < \varepsilon < c$), is better when innovation is licensed by a mean of a fixed fee or an auction with the same number of licenses. We also find that licensing the two firms by a mean of royalty or fixed fee is better for consumers than licensing only one firm. Finally, we show that no licensing or royalties licensing are the worst regimes for consumers.

$$SC(2F) = SC(2E) \quad > \quad SC(1F) = SC(1E) \quad > \quad SC(NL) \quad > \quad SC(royalties)$$

12. Optimal regime in terms of total surplus

Total surplus is higher when two firms are licensed by means of fixed fee or royalty. When innovation is not drastic compared with imitation, fixed fee licensing regime is better than auction. If innovation is drastic compared with imitation, than total surplus is higher two firms are licensed under auction regime. Finally, when there is no licensing, total surplus is lower (Appendix 25).

Conclusion

We tried in this paper to find the best licensing regimes for a patent holding firm in presence of imitation of a process innovation. We showed that Total revenue of the patent holding firm is optimal under royalties licensing regime for two firms when $\gamma < \varepsilon < 2\gamma + a - c$. For a drastic innovation compared with imitation, we found that the innovative firm has not interest in licensing and should become a monopoly.

Comparing optimal licensing regimes for the patent holding firm and optimal licensing regimes for consumers, we find that royalties licensing which is the best licensing strategy for the innovative firm is not the best strategy for consumers since the best strategy for them is fixed fee or auction. Consumer surplus and total surplus are higher when the licensing regime is by a mean of fixed fee for two firms or by a mean of an auction to two firms. While the worst licensing regime is no licensing or licensing by mean of a royalty. The second best licensing regime for the patent holding firm is an auction which is better than royalties regime for consumers and for total surplus.

Appendix 1

$$RT_1^{1F} - \pi_1^{NL} = \frac{(\varepsilon-\gamma)(-\varepsilon+2a+3\gamma-2c)}{8b} > 0 \text{ for } \gamma < \varepsilon < \frac{3\gamma+a-c}{2}$$

$$\begin{cases} \gamma < \varepsilon \Leftrightarrow \varepsilon - \gamma > 0 \\ -\varepsilon + 2a + 3\gamma - 2c = -(\varepsilon - 2(a-c) + 3\gamma) > 0 \forall \varepsilon \in \left[\gamma, \frac{3\gamma+a-c}{2}\right] \end{cases}$$

$$\Rightarrow \frac{(\varepsilon-\gamma)(-\varepsilon+2a+3\gamma-2c)}{8b} > 0$$

$$\Leftrightarrow RT_1^{1F} - \pi_1^{PL} > 0$$

Appendix 2

$$RT_1^{1F} - \pi_1^{PL} = \frac{7a^2 - 14ac + 14a\varepsilon + 7c^2 - 14c\varepsilon - 29\varepsilon^2 - 36\gamma^2 + 72\varepsilon\gamma}{72b}$$

$$= \frac{-29\varepsilon^2 + \varepsilon(14(a-c) + 72\gamma) + 7(a-c)^2 - 36\gamma^2}{72b}$$

Roots of $-29\varepsilon^2 + \varepsilon(14(a-c)+72\gamma) + 7(a-c)^2 - 36\gamma^2$ are

$\varepsilon' = \frac{7-6\sqrt{7}}{29}(a-c) + \frac{36-6\sqrt{7}}{29}\gamma$ and $\varepsilon'' = \frac{7+6\sqrt{7}}{29}(a-c) + \frac{36+6\sqrt{7}}{29}\gamma$

$\varepsilon' = -0{,}306(a-c) + 0{,}693\gamma$ and $\varepsilon'' = 0{,}788(a-c) + 1{,}788\gamma$

If $-0{,}306(a-c) + 0{,}693\gamma < \varepsilon < 0{,}788(a-c) + 1{,}788\gamma$ than

$-29\varepsilon^2 + \varepsilon(14(a-c) + 72\gamma) + 7(a-c)^2 - 36\gamma^2 > 0$

If $\varepsilon < -0{,}306(a-c) + 0{,}693\gamma$ or $\varepsilon > 0{,}788(a-c) + 1{,}788\gamma$ than

$-29\varepsilon^2 + \varepsilon(14(a-c) + 72\gamma) + 7(a-c)^2 - 36\gamma^2 < 0$

When $\frac{3\gamma+a-c}{2} < \varepsilon < 2\gamma + a - c$, the polynomial

$-29\varepsilon^2 + \varepsilon(14(a-c) + 72\gamma) + 7(a-c)^2 - 36\gamma^2$ has two signs :

If $\frac{3\gamma+a-c}{2} < \varepsilon < 1{,}788\gamma + 0{,}788(a-c)$ than

$-29\varepsilon^2 + \varepsilon(14(a-c) + 72\gamma) + 7(a-c)^2 - 36\gamma^2 > 0$ means $RT_1^{1F} - \pi_1^{PL} > 0$

If $1,788\gamma + 0,788(a-c) < \varepsilon < 2\gamma + a - c$ than
$-29\varepsilon^2 + \varepsilon(14(a-c) + 72\gamma) + 7(a-c)^2 - 36\gamma^2 < 0$ means $RT_1^{1F} - \pi_1^{PL} < 0$

Appendix 3

$$RT_1^{2F} - \pi_1^{PL} = \frac{(\varepsilon-\gamma)(-2\varepsilon+3\gamma+a-c)}{4b} > 0 \text{ when } \gamma < \varepsilon < \frac{3\gamma+a-c}{2}$$

$$RT_1^{2F} - \pi_1^{PL} = \frac{(\varepsilon-\gamma)(-2\varepsilon+3\gamma+a-c)}{4b} < 0 \text{ when } \frac{3\gamma+a-c}{2} < \varepsilon < 2\gamma+a-c$$

$$\begin{cases} \gamma < \varepsilon \Leftrightarrow \varepsilon - \gamma > 0 \\ -2\varepsilon+3\gamma+a-c = -(2\varepsilon-3\gamma-(a-c)) > 0 \forall \varepsilon \in \left[\gamma, \frac{3\gamma+a-c}{2}\right] \\ -2\varepsilon+3\gamma+a-c = -(2\varepsilon-3\gamma-(a-c)) < 0 \forall \varepsilon \in \left[\frac{3\gamma+a-c}{2}, 2\gamma+a-c\right] \end{cases}$$

Appendix 4

$$RT_1^{2F} = \frac{a^2 - 2ac + 10a\varepsilon + c^2 - 10c\varepsilon + \varepsilon^2 - 8a\gamma + 8c\gamma - 8\gamma^2 + 8\gamma\varepsilon}{16b}$$

$$RT_1^{1F} = \frac{a^2 - 2ac + 10a\varepsilon - 8a\gamma + c^2 - 10c\varepsilon + 8c\gamma + 7\varepsilon^2 - 4\varepsilon\gamma - 2\gamma^2}{16b}$$

$$RT_1^{2F} - RT_1^{1F} = -\frac{(\varepsilon-\gamma)^2}{8b} < 0$$

Appendix 5

$$RT_1^{2F} = \frac{a^2 - 2ac + 10a\varepsilon + c^2 - 10c\varepsilon + \varepsilon^2 - 8a\gamma + 8c\gamma - 8\gamma^2 + 8\gamma\varepsilon}{16b}$$

$$RT_1^{1F} = \frac{23a^2 - 46ac + 82a\varepsilon + 23c^2 - 82c\varepsilon + 23\varepsilon^2 - 36a\gamma + 36c\gamma - 36\gamma^2 + 36\varepsilon\gamma}{144b}$$

$$RT_1^{2F} - RT_1^{1F} = \frac{-7\varepsilon^2 + \varepsilon(4(a-c) + 18\gamma) - 18\gamma(a-c+\gamma) - 7(a-c)^2}{72b}$$

Delta of the numerator is

$$\Delta = (4(a-c)+18\gamma)^2 - 504\gamma(a-c+\gamma) - 196(a-c)^2 = -180(a-c)^2 - 360\gamma(a-c) - 180\gamma^2 < 0$$

Polynomial $-7\varepsilon^2+\varepsilon(4(a-c)+18\gamma)-18\gamma(a-c+\gamma)-7(a-c)^2$ has no roots. for $\varepsilon=0$

$-7\varepsilon^2+\varepsilon(4(a-c)+18\gamma)-18\gamma(a-c+\gamma)-7(a-c)^2=-18\gamma(a-c+\gamma)-7(a-c)^2<0$

then $RT_1^{\ 2F}-RT_1^{\ 1F}=\dfrac{-7\varepsilon^2+\varepsilon(4(a-c)+18\gamma)-18\gamma(a-c+\gamma)-7(a-c)^2}{72b}<0$

Appendix 6

$$RT_1^{\ 2F}=\frac{3(a-c+\varepsilon)^2}{16b}$$

$$RT_1^{\ 1F}=\frac{2(a-c+\varepsilon)^2}{9b}$$

$$RT_1^{\ 2F}-RT_1^{\ 1F}=-\frac{5(a-c+\varepsilon)^2}{144b}<0$$

Appendix 7

$$SC^{2F}-SC^{1F}=\frac{(\varepsilon-\gamma)(5\varepsilon+6(a-c)+\gamma)}{32b}>0$$

$$SC^{2F}-SC^{PL}=\frac{(\varepsilon-\gamma)(2\varepsilon+3(a-c)+\gamma)}{8b}>0$$

$$SC^{1F}-SC^{PL}=\frac{3(\varepsilon-\gamma)(\varepsilon+2(a-c)+\gamma)}{32b}>0$$

$$ST^{2F}-ST^{1F}=\frac{(\varepsilon-\gamma)(-19\varepsilon+14(a-c)+33\gamma)}{32b}>0$$

$$ST^{2F}-ST^{PL}=\frac{(\varepsilon-\gamma)(-2\varepsilon+9(a-c)+11\gamma)}{8b}>0$$

$$ST^{1F}-ST^{PL}=\frac{11(\varepsilon-\gamma)(\varepsilon+2(a-c)+\gamma)}{32b}>0$$

Appendix 8

$$SC^{2F}-SC^{1F}=\frac{17(a-c+\varepsilon)^2}{288b}>0$$

$$SC^{2F}-SC^{PL}=\frac{(\varepsilon-\gamma)(2\varepsilon+3(a-c)+\gamma)}{8b}>0$$

$$SC^{1F}-SC^{PL}=\frac{(5\varepsilon-(a-c)-6\gamma)(11\varepsilon+17(a-c)+6\gamma)}{288b}>0$$

$$ST^{2F} - ST^{1F} = \frac{-7\varepsilon^2 - 58c\varepsilon + 58a\varepsilon + 72\gamma\varepsilon - 7c^2 - 7a^2 - 72a\gamma + 72c\gamma - 72\gamma^2 + 14ac}{288b}$$

Roots of the numerator are:

$$\varepsilon' = \left(\frac{29 - 6\sqrt{21}}{7}\right)(a-c) + \left(\frac{36 - 6\sqrt{21}}{7}\right)\gamma \text{ and } \varepsilon'' = \left(\frac{29 + 6\sqrt{21}}{7}\right)(a-c) + \left(\frac{36 + 6\sqrt{21}}{7}\right)\gamma$$

since $\varepsilon' < \frac{3\gamma + a - c}{2} < 2\gamma + a - c < \varepsilon''$ than for ε verifying $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$,

numerator is positive which means that $ST^{2F} - ST^{1F} > 0$

$$ST^{2F} - ST^{PL} = \frac{(\varepsilon - \gamma)(-2\varepsilon + 9(a-c) + 11\gamma)}{8b} > 0$$

$$ST^{1F} - ST^{PL} = \frac{-65\varepsilon^2 - 266c\varepsilon + 266a\varepsilon + 396\gamma\varepsilon + 7c^2 + 7a^2 - 252a\gamma + 252c\gamma - 324\gamma^2 - 14ac}{288b}$$

Roots of the numerator are:

$$\varepsilon' = \left(\frac{133 - 36\sqrt{14}}{65}\right)(a-c) + \left(\frac{198 - 36\sqrt{14}}{65}\right)\gamma$$

and $\varepsilon'' = \left(\frac{133 + 36\sqrt{14}}{65}\right)(a-c) + \left(\frac{198 + 36\sqrt{14}}{65}\right)\gamma$

Since $\varepsilon' < \frac{3\gamma + a - c}{2} < 2\gamma + a - c < \varepsilon''$ than for ε verifying $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$,

numerator is positive which means that $ST^{1F} - ST^{PL} > 0$

Appendix 9

$$SC^{2F} - SC^{1F} = \frac{17(a - c + \varepsilon)^2}{288b} > 0$$

$$SC^{2F} - SC^{PL} = \frac{5(a - c + \varepsilon)^2}{32b} > 0$$

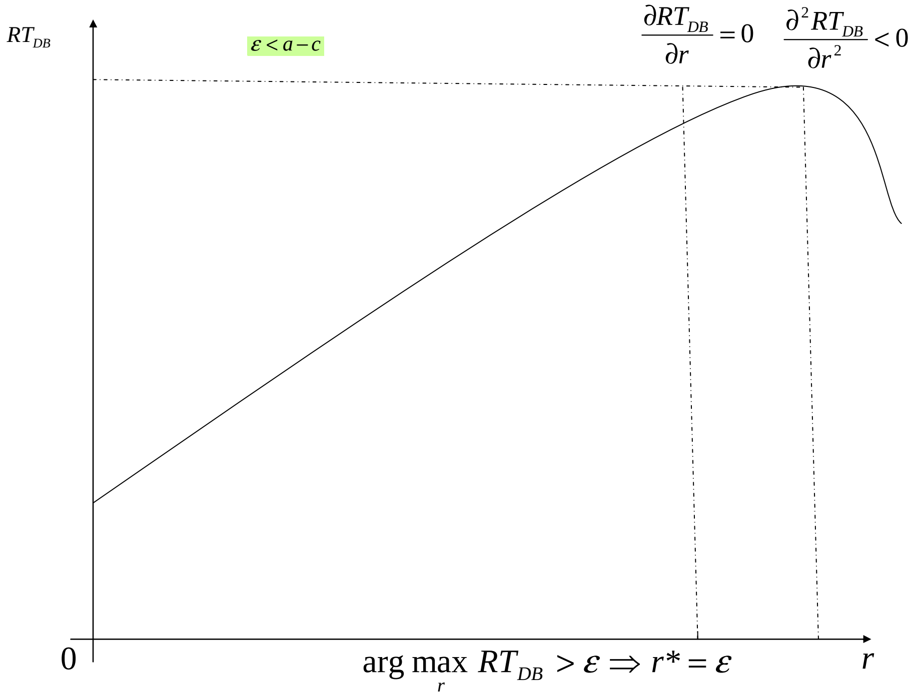
$$SC^{1F} - SC^{PL} = \frac{7(a - c + \varepsilon)^2}{72b} > 0$$

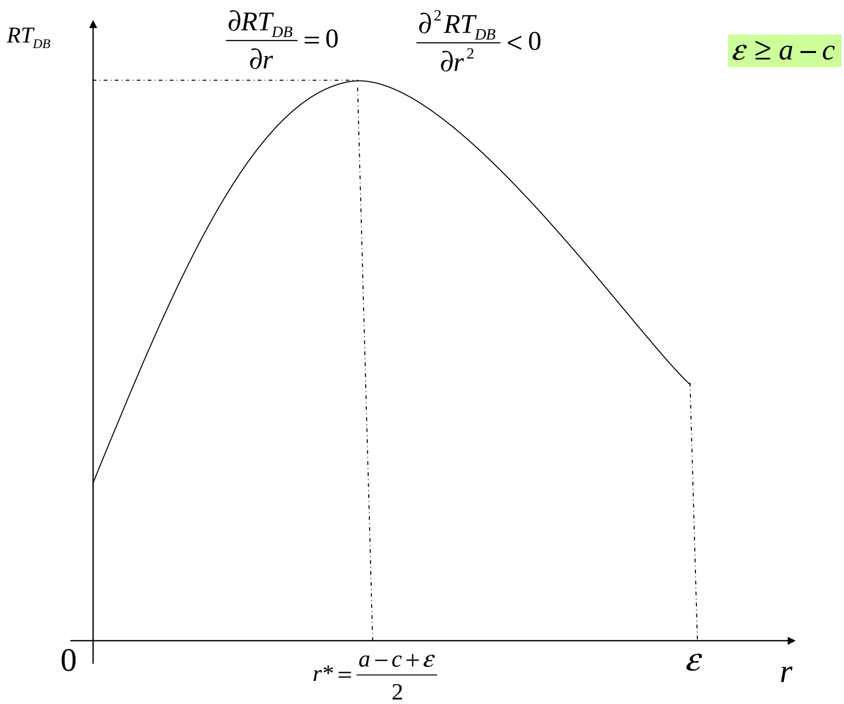
$$ST^{2F} - ST^{1F} = \frac{11(a - c + \varepsilon)^2}{288b} > 0$$

$$ST^{2F} - ST^{PL} = \frac{7(a - c + \varepsilon)^2}{32b} > 0$$

$$ST^{1F} - ST^{PL} = \frac{13(a-c+\varepsilon)^2}{72b} > 0$$

Appendix 10





Appendix 11

$$RT_{2\,firms}^{royalties} - RT^{NL} = \frac{-2\varepsilon^2 + \varepsilon(2(a-c)+3\gamma) + (a-c)\gamma - \gamma^2}{4b}$$

Numerator roots are:

$$\varepsilon' = \frac{3}{4}\gamma + \frac{1}{2}(a-c) - \frac{1}{4}\sqrt{\gamma^2 + 20\gamma(a-c) + 4(a-c)^2} \text{ and}$$

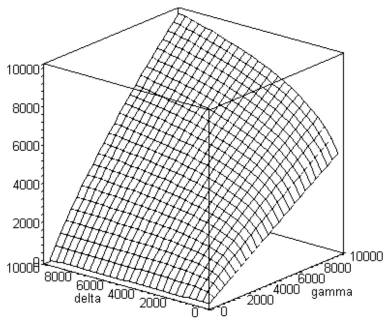
$$\varepsilon'' = \frac{3}{4}\gamma + \frac{1}{2}(a-c) + \frac{1}{4}\sqrt{\gamma^2 + 20\gamma(a-c) + 4(a-c)^2}$$

We can check that $\varepsilon' < \gamma < a-c < \varepsilon''$.

In fact, $\gamma - \varepsilon' = \frac{1}{4}\gamma - \frac{1}{2}(a-c) + \frac{1}{4}\sqrt{\gamma^2 + 20\gamma(a-c) + 4(a-c)^2} > 0$ since

$\gamma + \sqrt{\gamma^2 + 20\gamma(a-c) + 4(a-c)^2} > 2(a-c)$.

Plotting the function $\gamma - \varepsilon'$ depending on variables γ and $\delta = a-c$, we can see that $\gamma - \varepsilon'$ is always positive



$$\varepsilon'' - (a-c) = \frac{3}{4}\gamma - \frac{1}{2}(a-c) + \frac{1}{4}\sqrt{\gamma^2 + 20\gamma(a-c) + 4(a-c)^2} > \gamma - \varepsilon' > 0$$

Which means that numerator sign is positive

$$RT_{2\,firms}^{royalties} - RT^{NL} = \frac{-2\varepsilon^2 + \varepsilon(2(a-c)+3\gamma) + (a-c)\gamma - \gamma^2}{4b} > 0$$

Appendix 12

$RT_{2\,firms}^{royalties} = RT^{NL} = \dfrac{(a-c+\varepsilon)^2}{4b}$ (using equations (37) and (6))

$$SC_{2\,firms}^{royalties} = SC^{NL} = \frac{(a-c+\varepsilon)^2}{8b}$$

$$ST_{2\,firms}^{royalties} = ST^{NL} = \frac{3(a-c+\varepsilon)^2}{8b}$$

Appendix 13

$\pi_1 + E - \pi_1^{NL} = \dfrac{(\varepsilon-\gamma)(6(a-c)-5\varepsilon+11\gamma)}{16b} > 0$ since for $\varepsilon < \dfrac{3\gamma + a - c}{2}$ we have $6(a-c)-5\varepsilon+11\gamma > 0$

Appendix 14

$$\pi_1 + E - \pi_1^{NL} = \frac{2(a-c+\varepsilon)^2}{9b} - \frac{(a-c+3\varepsilon-2\gamma)^2}{16b}$$

$$= \frac{23(a-c)^2 + (10\varepsilon+36\gamma)(a-c) + 108\varepsilon\gamma - 49\varepsilon^2 - 36\gamma^2}{144b}$$

Numerator roots are:

$\varepsilon' = \dfrac{5+24\sqrt{2}}{49}(a-c) + \dfrac{54+24\sqrt{2}}{49}\gamma$ and $\varepsilon'' = \dfrac{5-24\sqrt{2}}{49}(a-c) + \dfrac{54-24\sqrt{2}}{49}\gamma$

$\varepsilon' = 0{,}7947(a-c) + 1{,}794\gamma$ and $\varepsilon'' = -0{,}5906(a-c) + 0{,}4093\gamma$

Numerator is always positive when $\varepsilon' < \varepsilon < \varepsilon''$

Appendix 15

$$\pi_1 + 2E - \pi_1^{NL} = \frac{(\varepsilon-\gamma)(4(a-c)-7\varepsilon+11\gamma)}{8b} > 0$$

Appendix 16

$$\pi_1 + 2E - \pi_1^{NL} = \frac{3(a-c+\varepsilon)^2}{16b} - \frac{(a-c+3\varepsilon-2\gamma)^2}{16b}$$

$$= \frac{(a-c)^2 + 2\gamma(a-c) + 6\varepsilon\gamma - 3\varepsilon^2 - 2\gamma^2}{8b}$$

Roots of the numerator are: $\varepsilon' = \left(1 - \dfrac{\sqrt{3}}{3}\right)\gamma - \dfrac{\sqrt{3}}{3}(a-c)$ and $\varepsilon'' = \left(1 + \dfrac{\sqrt{3}}{3}\right)\gamma + \dfrac{\sqrt{3}}{3}(a-c)$

$\varepsilon' = 0{,}4226\gamma - 0{,}5773(a-c)$ and $\varepsilon'' = 1{,}5773\gamma + 0{,}5773(a-c)$

Numerator is always positive when $\varepsilon' < \varepsilon < \varepsilon''$

Appendix 17

a) If $\varepsilon < \frac{3\gamma + a - c}{2}$

Comparing equations of total revenue (49) and (43) of patent holding firm, we find

$$RT_1^{2E} - RT_1^{1E} = \frac{(\varepsilon - \gamma)(2(a-c) - 9\varepsilon + 11\gamma)}{16b}$$

If $\gamma < \varepsilon < \frac{11\gamma + 2(a-c)}{9}$ then $RT_1^{2E} - RT_1^{1E} > 0$

If $\frac{11\gamma + 2(a-c)}{9} < \varepsilon < \frac{3\gamma + a - c}{2}$ then $RT_1^{2E} - RT_1^{1E} < 0$

Patent holding firm profit is better when licensing two firms than licensing only one firm when $\gamma < \varepsilon < \frac{11\gamma + 2(a-c)}{9}$. when $\frac{11\gamma + 2(a-c)}{9} < \varepsilon < \frac{3\gamma + a - c}{2}$ licensing just one firm is better than licensing two firms.

b) If $\varepsilon > \frac{3\gamma + a - c}{2}$

Comparing equations of total revenue (51) and (45) of patent holding firm, we find

$$RT_1^{2E} - RT_1^{1E} = -\frac{5(a - c + \varepsilon)^2}{144b} < 0.$$

Licensing just one firm is better than licensing two firms when innovation is drastic compared with imitation.

<u>Appendix 18</u>

In the auction licensing to only one firm:

- If $\varepsilon < \frac{3\gamma + a - c}{2}$

Consumer surplus $SC = \frac{(3(a-c) + 2\varepsilon + \gamma)^2}{32b}$

Total surplus $ST = \frac{15(a-c)^2 + 20(a-c)\varepsilon + 28\varepsilon^2 + 10(a-c)\gamma - 36\varepsilon\gamma + 23\gamma^2}{32b}$

- If $\varepsilon > \frac{3\gamma + a - c}{2}$

Consumer surplus $SC = \frac{(a - c + \varepsilon)^2}{9b}$

Total surplus $ST = \frac{4(a-c+\varepsilon)^2}{9b}$

In the auction licensing to two firms:

- If $\varepsilon < \frac{3\gamma + a - c}{2}$

Consumer surplus $SC = \frac{9(a-c+\varepsilon)^2}{32b}$

Total surplus $ST = \frac{3\left(5(a-c)^2 + 18(a-c)\varepsilon - 8(a-c)\gamma + \varepsilon^2 + 16\varepsilon\gamma - 12\gamma^2\right)}{32b}$

- If $\varepsilon > \frac{3\gamma + a - c}{2}$

Consumer surplus $SC = \frac{9(a-c+\varepsilon)^2}{32b}$

Total surplus $ST = \frac{19(a-c+\varepsilon)^2}{32b}$

Comparison of consumer surplus and total surplus

- If $\varepsilon < \frac{3\gamma + a - c}{2}$

$$SC_{1E} - SC_{NL} = \frac{3(\varepsilon - \gamma)(2(a-c) + \varepsilon + \gamma)}{32b} > 0$$

$$SC_{2E} - SC_{NL} = \frac{(\varepsilon - \gamma)(3(a-c) + 2\varepsilon + \gamma)}{32b} > 0$$

$$SC_{2E} - SC_{1E} = \frac{(\varepsilon - \gamma)(6(a-c) + 5\varepsilon + \gamma)}{32b} > 0$$

$$ST_{1E} - ST_{NL} = \frac{5(\varepsilon - \gamma)(2(a-c) + \varepsilon + \gamma)}{32b} > 0$$

$ST_{2E} - ST_{NL} = \frac{(\varepsilon - \gamma)(11(a-c) - 5\varepsilon + 16\gamma)}{8b} > 0$ since $11(a-c) - 5\varepsilon + 16\gamma > 0$ when $\varepsilon < \frac{3\gamma + a - c}{2}$

$$ST_{2E}-ST_{1E}=\frac{(\varepsilon-\gamma)(34(a-c)-25\varepsilon+59\gamma)}{32b}>0 \text{ since } 34(a-c)-25\varepsilon+59\gamma>0 \text{ when}$$

$$\varepsilon<\frac{3\gamma+a-c}{2}$$

- If $\frac{3\gamma+a-c}{2}<\varepsilon<2\gamma+a-c$

$$SC_{1E}-SC_{NL}=\frac{-(a-c-5\varepsilon+6\gamma)(17(a-c)+11\varepsilon+6\gamma)}{288b}>0$$

In fact $a-c-5\varepsilon+6\gamma<0$ when $\frac{3\gamma+a-c}{2}<\varepsilon<2\gamma+a-c$

$$SC_{2E}-SC_{NL}=\frac{(\varepsilon-\gamma)(3(a-c)+2\varepsilon+\gamma)}{32b}>0$$

$$SC_{2E}-SC_{1E}=\frac{17(a-c+\varepsilon)^2}{288b}>0$$

$$ST_{1E}-ST_{NL}=\frac{-7(a-c)^2+(166\varepsilon-180\gamma)(a-c)-79\varepsilon^2+324\varepsilon\gamma-252\gamma^2}{288b}>0$$

In fact, numerator roots are:

$$\varepsilon'=\left(\frac{162-24\sqrt{11}}{79}\right)\gamma-\left(\frac{83-24\sqrt{11}}{79}\right)(a-c) \text{ and } \varepsilon''=\left(\frac{162+24\sqrt{11}}{79}\right)\gamma+\left(\frac{83+24\sqrt{11}}{79}\right)(a-c)$$

Which is equivalent to $\varepsilon'=1{,}043\gamma-0{,}043(a-c)$ and $\varepsilon''=3{,}0582\gamma+2{,}0582(a-c)$

Numerator is always positive when $\varepsilon'<\varepsilon<\varepsilon''$.

Since $\varepsilon'<\frac{3\gamma+a-c}{2}<\varepsilon<2\gamma+a-c<\varepsilon''$ than $ST_{1E}-ST_{PL}>0$

$$ST_{2E}-ST_{NL}=\frac{(a-c)^2+(a-c)(7\varepsilon-5\gamma)-\varepsilon^2+9\varepsilon\gamma-7\gamma^2}{8b}>0$$

In fact, $ST_{2E}-ST_{NL}=\frac{(a-c)^2+(a-c)(7\varepsilon-5\gamma)-\varepsilon^2+9\varepsilon\gamma-7\gamma^2}{8b}$

Numerator roots are:

$$\varepsilon'=\left(\frac{9-\sqrt{53}}{2}\right)\gamma-\left(\frac{7-\sqrt{53}}{2}\right)(a-c) \text{ and } \varepsilon''=\left(\frac{9+\sqrt{53}}{2}\right)\gamma+\left(\frac{7+\sqrt{53}}{2}\right)(a-c)$$

Which is equivalent to $\varepsilon'=0{,}8599\gamma-0{,}14(a-c)$ and $\varepsilon''=8{,}14\gamma+7{,}14(a-c)$

Numerator is always positive when $\varepsilon'<\varepsilon<\varepsilon''$.

Since $\varepsilon'<\frac{3\gamma+a-c}{2}<\varepsilon<2\gamma+a-c<\varepsilon''$ than $ST_{2E}-ST_{PL}>0$

$$ST_{2E} - ST_{1E} = \frac{43(a-c+\varepsilon)^2}{288b} > 0$$

- If $\varepsilon \geq 2\gamma + a - c$

$$SC_{1E} - SC_{NL} = \frac{7(a-c+\varepsilon)^2}{72b} > 0$$

$$SC_{2E} - SC_{PL} = \frac{5(a-c+\varepsilon)^2}{32b} > 0$$

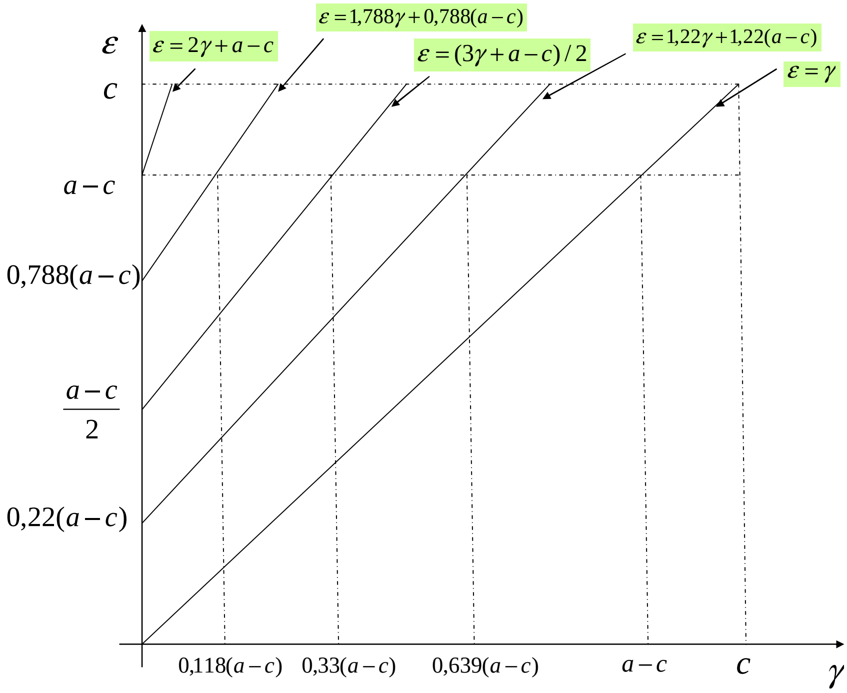
$$SC_{2E} - SC_{1E} = \frac{17(a-c+\varepsilon)^2}{288b} > 0$$

$$ST_{1E} - ST_{NL} = \frac{5(a-c+\varepsilon)^2}{72b} > 0$$

$$ST_{2E} - ST_{NL} = \frac{7(a-c+\varepsilon)^2}{32b} > 0$$

$$ST_{2E} - ST_{1E} = \frac{43(a-c+\varepsilon)^2}{288b} > 0$$

Appendix 19



$$RT_1^{2F} - RT_1^{NL} = \frac{(\varepsilon-\gamma)(-2\varepsilon+3\gamma+a-c)}{4b} > 0$$

$$RT_1^{1F} - RT_1^{2F} = \frac{3(\varepsilon-\gamma)^2}{8b} > 0$$

$$RT_1^{1E} - RT_1^{1F} = \frac{(\varepsilon-\gamma)(-3\varepsilon+2(a-c)+5\gamma)}{16b} > 0$$

$$RT_1^{2E} - RT_1^{1E} = \frac{(\varepsilon-\gamma)(-9\varepsilon+2(a-c)+11\gamma)}{16b} > 0$$

If $\varepsilon < a-c$, $RT_1^{royalties} - RT_1^{2E} = \frac{3\left(\varepsilon^2 - 4\varepsilon\gamma + 2\gamma(a-c) + 3\gamma^2\right)}{8b} > 0$

Numerator has no roots since $\Delta = 4\gamma(\gamma - 2(a-c)) < 0$. for $\varepsilon = 0$ numerator sign is $sign\left(2\gamma(a-c) + 3\gamma^2\right) > 0$

We can check that for $\gamma < \varepsilon < \frac{11\gamma + 2(a-c)}{9}$ and $\varepsilon < a-c$, we have $\gamma < a-c < 2(a-c)$

If $\varepsilon \geq a-c$, $RT_1^{royalties} - RT_1^{2E} = \frac{3\left((a-c)^2 + (4\gamma - 2\varepsilon)(a-c) + 3\varepsilon^2 - 8\varepsilon\gamma + 6\gamma^2\right)}{16b} > 0$

Numerator has no roots since $\Delta = -(8\gamma^2 + 16\gamma(a-c) + 8(a-c)^2) < 0$. for $\varepsilon = 0$ the numerator sign is $sign\left((a-c)^2 + (4\gamma)(a-c) + 6\gamma^2\right) > 0$

Appendix 20

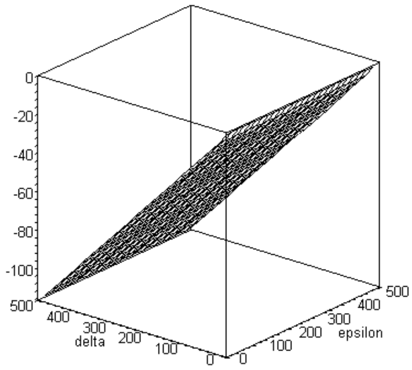
If $\varepsilon < a-c$, $RT_1^{royalties} - RT_1^{1E} = \dfrac{2(a-c)\varepsilon - 3\varepsilon^2 + 10(a-c)\gamma - 4\varepsilon\gamma + 7\gamma^2}{16b} > 0$

In fact, numerator roots are:

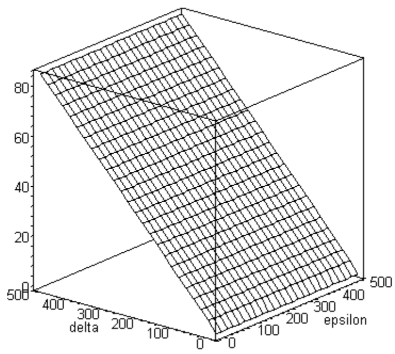
$\varepsilon' = \dfrac{a-c-2\gamma}{3} - \dfrac{1}{3}\sqrt{(a-c+\gamma)(a-c+25\gamma)}$ and $\varepsilon'' = \dfrac{a-c-2\gamma}{3} + \dfrac{1}{3}\sqrt{(a-c+\gamma)(a-c+25\gamma)}$

We can check that $\varepsilon' < \dfrac{11\gamma + 2(a-c)}{9} < \dfrac{3\gamma + a - c}{2} < \varepsilon''$ which means that $RT_1^{royalties} - RT_1^{1E}$ is positive.

Denoting by $\delta = a-c$, $\varepsilon' - \dfrac{11\gamma + 2(a-c)}{9} = \dfrac{\delta - 17\gamma}{9} - \dfrac{1}{3}\sqrt{(\delta+\gamma)(\delta+25\gamma)} < 0$



$\varepsilon'' - \dfrac{3\gamma + a - c}{2} = -\dfrac{\delta + 13\gamma}{6} + \dfrac{1}{3}\sqrt{(\delta+\gamma)(\delta+25\gamma)} > 0$



Si $\varepsilon > a-c$, $RT_1^{royalties} - RT_1^{1E} = \frac{(a-c+\gamma)(-4\varepsilon+7\gamma+3(a-c))}{16b} > 0$

Appendix 21

$$RT_1^{1E} - RT_1^{2E} = \frac{(\varepsilon-\gamma)(9\varepsilon-11\gamma-2(a-c))}{16b} > 0$$

$$RT_1^{2E} - RT_1^{1F} = \frac{(\varepsilon-\gamma)(-3\varepsilon+a-c+4\gamma)}{4b}$$

$$RT_1^{2E} - RT_1^{1F} > 0 \text{ if } \frac{11\gamma+2(a-c)}{9} < \varepsilon < \frac{4\gamma+a-c}{3}$$

$$RT_1^{2E} - RT_1^{1F} < 0 \text{ if } \frac{4\gamma+a-c}{3} < \varepsilon < \frac{3\gamma+a-c}{2}$$

$$RT_1^{1F} - RT_1^{2F} = \frac{3(\varepsilon-\gamma)^2}{8b} > 0$$

$$RT_1^{2F} - RT_1^{PL} = \frac{(\varepsilon-\gamma)(-2\varepsilon+a-c+3\gamma)}{4b} > 0$$

Appendix 22

$$RT_1^{2E} - RT_1^{1F} = \frac{((a-c)^2-7\varepsilon(a-c)+\varepsilon^2+9\gamma(a-c)+9\gamma^2-9\varepsilon\gamma)}{36b} < 0$$

In fact, numerator roots are:

$$\varepsilon' = \frac{9-3\sqrt{5}}{2}\gamma + \frac{7-3\sqrt{5}}{2}(a-c) \text{ and } \varepsilon'' = \frac{9+3\sqrt{5}}{2}\gamma + \frac{7+3\sqrt{5}}{2}(a-c)$$

Since $\varepsilon' < \frac{3\gamma+a-c}{2} < 1{,}788\gamma + 0{,}788(a-c) < \varepsilon''$, than $RT_1^{2E} - RT_1^{1F}$ is non positive.

$$RT_1^{1E} - RT_1^{1F} = \frac{(a-c-\varepsilon+2\gamma)^2}{16b} > 0$$

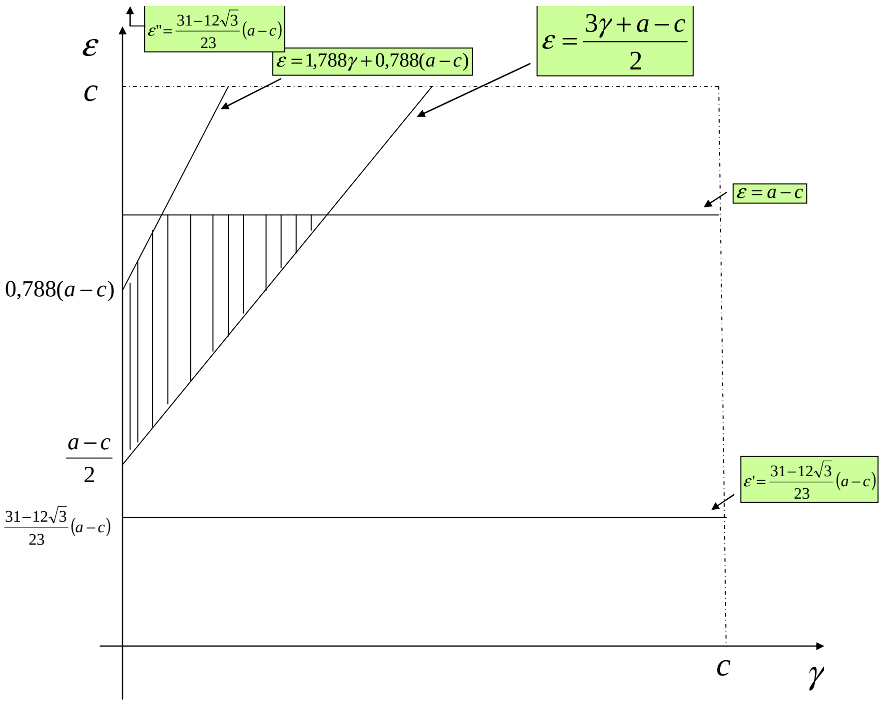
$RT_1^{2E} - RT_1^{PL} = \frac{((a-c)^2-3\varepsilon^2+2\gamma(a-c)-2\gamma^2-6\varepsilon\gamma)}{8b} > 0$ (see proof in auction licensing section)

$$RT_1^{PL} - RT_1^{2F} = \frac{(\varepsilon-\gamma)(2\varepsilon-3\gamma-(a-c))}{4b} > 0$$

If $\varepsilon < a - c$, $RT_1^{royalties} - RT_1^{1E} = \dfrac{-23(a-c)^2 + 62\varepsilon(a-c) - 23\varepsilon^2}{144b} > 0$

In fact, numerator roots are: $\varepsilon' = \dfrac{31 - 12\sqrt{3}}{23}(a-c)$ and $\varepsilon'' = \dfrac{31 + 12\sqrt{3}}{23}(a-c)$

Since $\dfrac{3\gamma + a - c}{2} < \varepsilon < 1{,}788\gamma + 0{,}788(a-c)$ and $\varepsilon < a - c$ than the numerator is positive and $RT_1^{royalties} > RT_1^{1E}$



Si $\varepsilon \geq a - c$, $RT_1^{royalties} - RT_1^{1E} = \dfrac{(a - c + \varepsilon)^2}{36b} > 0$

Appendix 23

If $\varepsilon < a - c$, $RT_1^{royalties} - RT_1^{1E} = \dfrac{-23(a-c)^2 + 62\varepsilon(a-c) - 23\varepsilon^2}{144b} > 0$

If $\varepsilon \geq a - c$, $RT_1^{royalties} - RT_1^{1E} = \dfrac{(a - c + \varepsilon)^2}{36b} > 0$

$RT_1^{1E} - RT_1^{2E} = \frac{5(a-c+\varepsilon)}{144b} > 0$

$RT_1^{2E} - RT_1^{PL} = \frac{\left((a-c)^2 + 2\gamma(a-c) + 6\varepsilon\gamma - 2\gamma^2 - 3\varepsilon^2\right)}{8b} > 0$ (See proof in auction licensing section)

$RT_1^{PL} - RT_1^{1F} = \frac{29\varepsilon^2 - \varepsilon(14(a-c) + 72\gamma) - 7(a-c)^2 + 36\gamma^2}{72b} > 0$ (See proof in fixed fee licensing section)

$RT_1^{1F} - RT_1^{2F} = \frac{7\varepsilon^2 - \varepsilon(4(a-c) + 18\gamma) + 7(a-c)^2 + 18\gamma^2}{72b} > 0$ (See proof in fixed fee licensing section)

<u>Appendix 24</u>

when $2\gamma + a - c < \varepsilon < c$ we have $\varepsilon \geq a - c$ which means $RT_1^{royalties} - RT_1^{NL} = 0$

$RT_1^{NL} - RT_1^{1F} = \frac{(a-c+\varepsilon)^2}{36b} > 0$

$RT_1^{1F} - RT_1^{1E} = 0$

$RT_1^{1F} - RT_1^{2E} = \frac{5(a-c+\varepsilon)^2}{144b} > 0$

$RT_1^{2E} - RT_1^{2F} = 0$

<u>Appendix 25</u>

- If $\gamma < \varepsilon < \frac{3\gamma + a - c}{2}$

$ST^{2F} > ST^{2E} > ST^{1F} > ST^{1E} > ST^{PL}$

In fact,

$ST^{2F} - ST^{2E} = \frac{(\varepsilon - \gamma)(3\varepsilon - 5\gamma - 2(a-c))}{8b} > 0$

$ST^{2E} - ST^{1F} = \frac{(\varepsilon - \gamma)(-31\varepsilon + 53\gamma + 22(a-c))}{32b} > 0$

$$ST^{1F} - ST^{1E} = \frac{3(\varepsilon-\gamma)(\varepsilon+\gamma+2(a-c))}{16b} > 0$$

If $\varepsilon < a-c$, $ST^{royalties} - ST^{NL} = \frac{(-\gamma)(-9\varepsilon+7\gamma+5(a-c))}{8b}$

$ST^{royalties} - ST^{NL} < 0$ if $\varepsilon < \frac{7\gamma+5(a-c)}{9}$

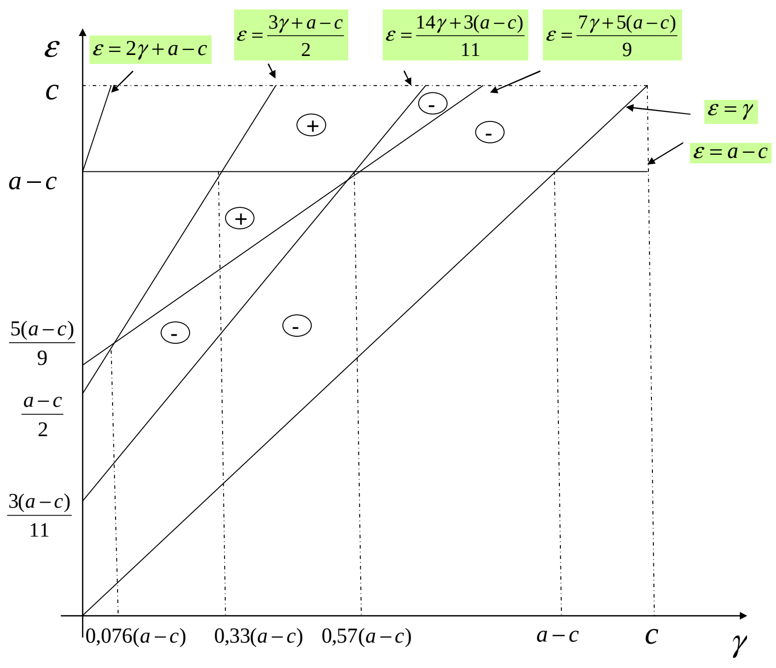
$ST^{royalties} - ST^{NL} > 0$ if $\varepsilon > \frac{7\gamma+5(a-c)}{9}$

If $\varepsilon \geq a-c$, $ST^{royalties} - ST^{NL} = \frac{(11\varepsilon-14\gamma-3(a-c))(-\varepsilon+2\gamma+a-c)}{32b}$

$ST^{royalties} - ST^{NL} < 0$ if $\varepsilon < \frac{14\gamma+3(a-c)}{11}$

$ST^{royalties} - ST^{NL} > 0$ if $\varepsilon > \frac{14\gamma+3(a-c)}{11}$

Sign of : $ST^{royalties} - ST^{NL}$



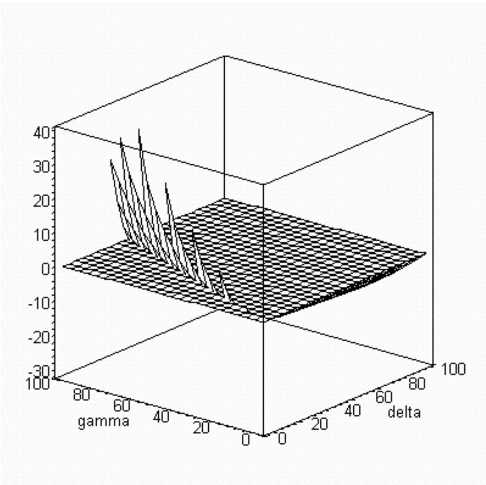
If $\varepsilon < a-c$, $ST^{royalties} - ST^{2F} = \frac{2\varepsilon^2-(9(a-c)+4\gamma)\varepsilon+4(a-c)\gamma+4\gamma^2}{8b} < 0$

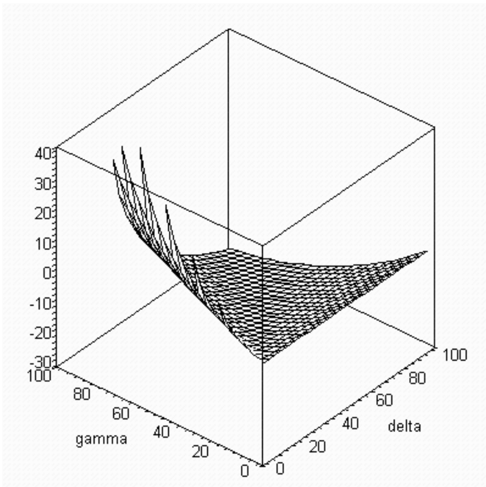
In fact,

The numerator roots are $\varepsilon' = \gamma + \frac{9}{4}(a-c) + \frac{1}{4}\sqrt{81(a-c)^2 + 40\gamma(a-c) - 16\gamma^2}$ and $\varepsilon' = \gamma + \frac{9}{4}(a-c) - \frac{1}{4}\sqrt{81(a-c)^2 + 40\gamma(a-c) - 16\gamma^2}$

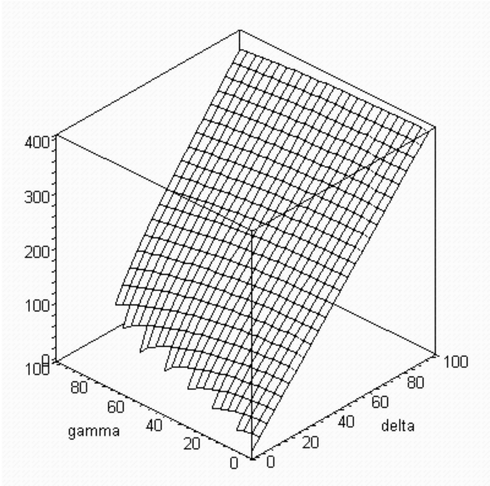
We can check that $\varepsilon' < \gamma < \frac{3\gamma + a - c}{2} < a - c < \varepsilon''$ and than numerator sign is non positive. In fact, denoting by $\delta = a - c$, we can write $\varepsilon' - \gamma = \frac{9}{4}\delta - \frac{1}{4}\sqrt{81\delta^2 + 40\gamma\delta - 16\gamma^2}$ **.**

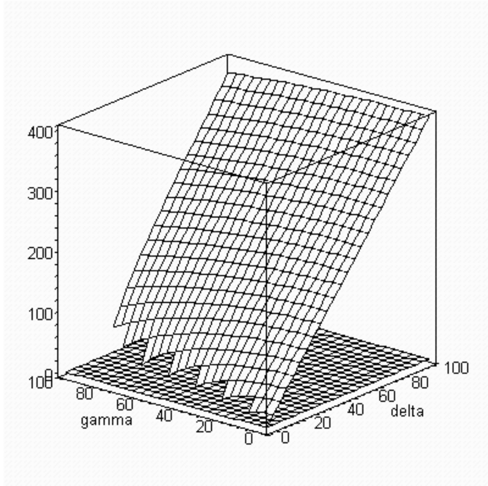
The following 3d plot shows that the sign of $\varepsilon' - \gamma$ is non positive for $\gamma < \delta$ (necessary condition because in this case we have $\gamma < \varepsilon < a - c$ since imitation ca not be greater than innovation and innovation can not exceed $a - c$)



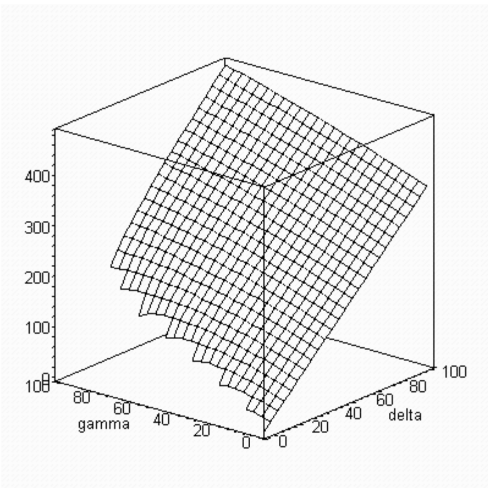


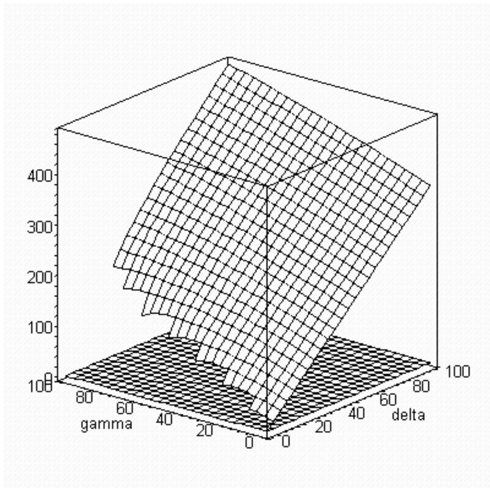
We also have $\varepsilon'' - \frac{3\gamma + a - c}{2} = -\frac{1}{2}\gamma + \frac{7}{4}\delta + \frac{1}{4}\sqrt{81\delta^2 + 40\gamma\delta - 16\gamma^2} > 0$. Plotting this equation in 3d we obtain the followings graphs:





$$\varepsilon'' - (a-c) = \gamma + \frac{5}{4}\delta + \frac{1}{4}\sqrt{81\delta^2 + 40\gamma\delta - 16\gamma^2} > 0$$





The licensing regime for total surplus when $\gamma < \varepsilon < \frac{3\gamma + a - c}{2}$ is fixed fee for two firms.

- If $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$

$ST^{2E} > ST^{2F} > ST^{1F} > ST^{1E} > ST^{royalties} > ST^{NL}$ or
$ST^{2E} > ST^{2F} > ST^{1F} > ST^{1E} > ST^{NL} > ST^{royalties}$

Royalties are better in terms of total surplus than no licensing when $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ and $\varepsilon > \frac{7\gamma + 5(a-c)}{9}$

While royalties licensing become the worst regime when $\frac{3\gamma + a - c}{2} < \varepsilon < \frac{7\gamma + 5(a-c)}{9}$ which corresponds to a weak imitation $\gamma < \frac{a-c}{13}$.

The best licensing regime for total surplus is an auction licensing for two firms

In fact,

$$ST^{2E} - ST^{2F} = \frac{(\varepsilon - 2\gamma - (a-c))^2}{8b} > 0$$

$ST^{2F} - ST^{1F} > 0$ (see proof in fixed fee licensing section)

$$ST^{1F} - ST^{1E} = -\frac{(\varepsilon + 6\gamma + 7(a-c))(7\varepsilon - 6\gamma + a - c)}{144b} > 0$$

$ST^{1E} - ST^{NL} > 0$ (see proof in auction licensing section)

If $\varepsilon < a - c$, $ST^{2E} - ST^{royalties} = \frac{\left((a-c)^2 + 7\varepsilon(a-c) - \varepsilon^2\right)}{8b} > 0$

In fact, numerator signs are: $\varepsilon' = \frac{7 - \sqrt{53}}{2}(a-c) < 0 < a - c < \varepsilon'' = \frac{7 + \sqrt{53}}{2}(a-c)$ and than numerator sign when $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ and $0 < \varepsilon < a - c$ is positive.

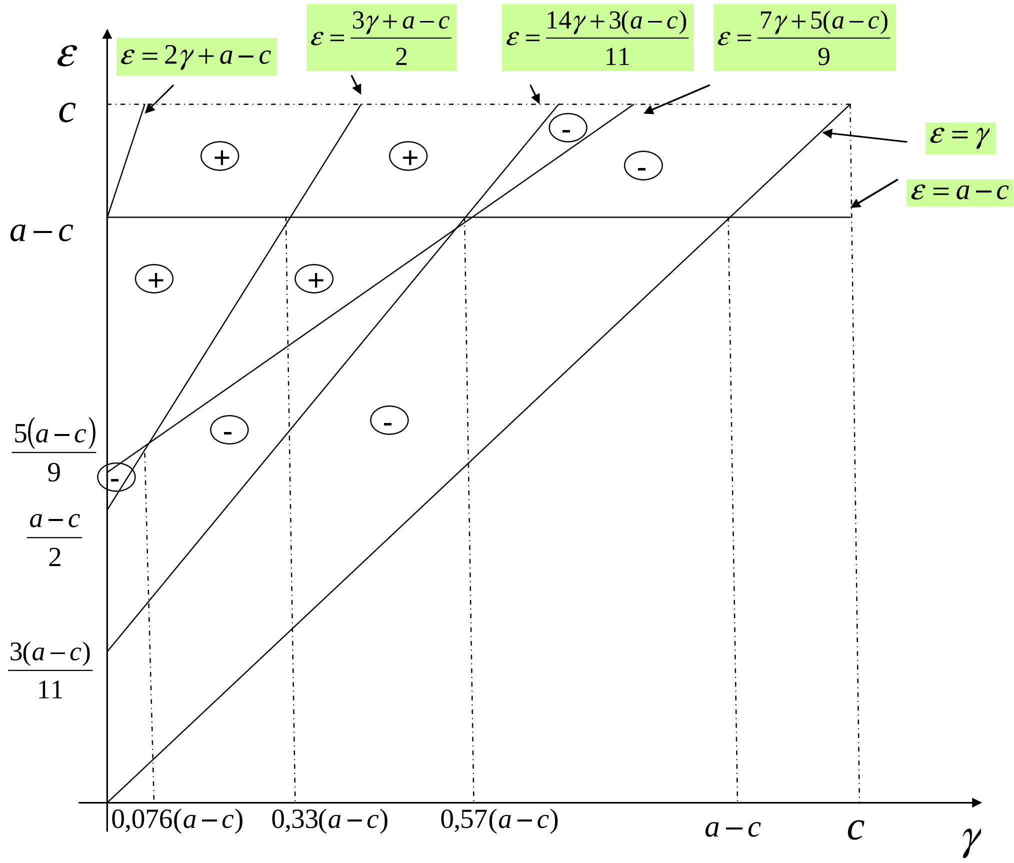
If $\varepsilon \geq a - c$, $ST^{2E} - ST^{royalties} = \frac{7(a - c + \varepsilon)^2}{32b} > 0$

If $\varepsilon \geq a - c$, $ST^{1E} - ST^{royalties} = \frac{5(a - c + \varepsilon)^2}{72b} > 0$

If $\varepsilon < a - c$, $ST^{1E} - ST^{royalties} = -\frac{7(a-c)^2 - 166\varepsilon(a-c) + 79\varepsilon^2}{288b} > 0$

In fact, numerator roots are: $\varepsilon' = \frac{83 - 24\sqrt{11}}{79}(a-c) < \frac{a-c}{2} < a - c < \varepsilon'' = \frac{83 + 24\sqrt{11}}{79}(a-c)$ and than numerator sign when $\frac{3\gamma + a - c}{2} < \varepsilon < 2\gamma + a - c$ and $\varepsilon < a - c$ is positive.

<u>Sign of</u> : $ST^{royalties} - ST^{NL}$



If $\varepsilon < a-c$, $ST^{royalties} - ST^{NL} = \dfrac{(-\gamma)(-9\varepsilon + 7\gamma + 5(a-c))}{8b}$

$ST^{royalties} - ST^{NL} < 0$ if $\varepsilon < \dfrac{7\gamma + 5(a-c)}{9}$

$ST^{royalties} - ST^{NL} > 0$ if $\varepsilon > \dfrac{7\gamma + 5(a-c)}{9}$

If $\varepsilon \geq a-c$, $ST^{royalties} - ST^{NL} = \dfrac{(11\varepsilon - 14\gamma - 3(a-c))(-\varepsilon + 2\gamma + a - c)}{32b} > 0$

- If $2\gamma + a - c < \varepsilon < c$

$ST^{2E} = ST^{2F} > ST^{1F} > ST^{1E} > ST^{NL} = ST^{royalties}$

In fact,

$ST^{2E} - ST^{2F} = 0$

$ST^{2F} - ST^{1F} > 0$ (See proof in fixed fee licensing section)

$$ST^{1F} - ST^{1E} = \frac{(a - c + \varepsilon)^2}{9b} > 0$$

$ST^{1E} - ST^{NL} > 0$ (See proof in auction licensing section)

$$ST^{royalties} - ST^{NL} = 0$$

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